DIFFERENTIAL ANALYSIS OF FLUID FLOW

A: Mathematical Formulation (4.1.1, 4.2, 6.1-6.4)

B: Inviscid Flow: Euler Equation/Some Basic, Plane Potential Flows (6.5-6.7)

C: Viscous Flow: Navier-Stokes Equation (6.8-6.10)

Introduction

Differential Analysis

- There are situations in which the details of the flow are important, e.g., pressure and shear stress variation along the wing….
- Therefore, we need to develop relationship that apply at a point or at least in a very small region (infinitesimal volume) with a given flow field.
- This approach is commonly referred to as differential analysis.
- The solutions of the equations are rather difficult.
- Computational Fluid Dynamic (CFD) can be applied to complex flow problems.

PART A

Mathematical Formulation

(Sections 4.1.1, 4.2, 6.1-6.4)

Fluid Kinematics (4.1.1, 4.2)

- Kinematics involves position, velocity and acceleration, not forces.
- kinematics of the motion:

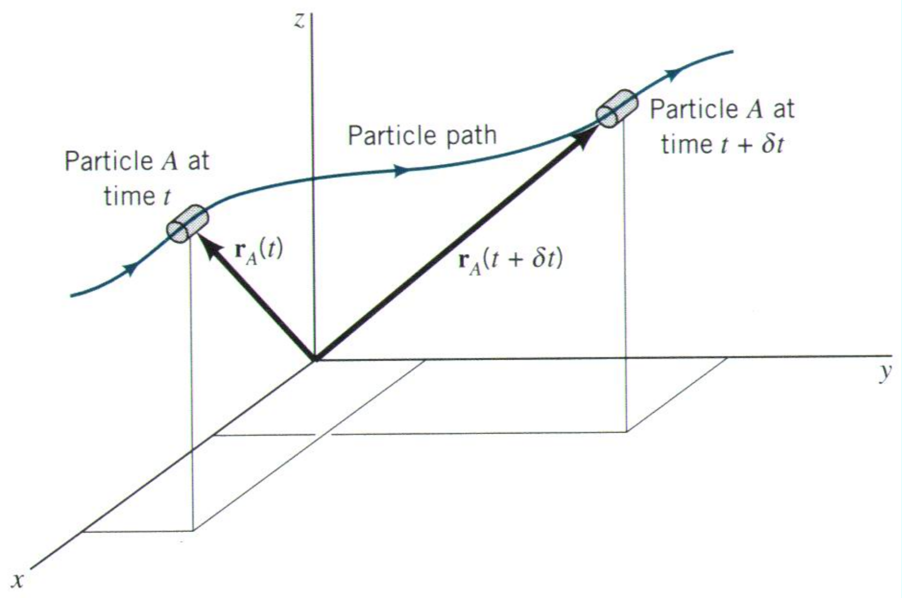
  velocity and acceleration of the fluid, and the description and visualization of its motion.
- The analysis of the specific force necessary to produce the motion - the dynamics of the motion.

4.1 The Velocity Field

A field representation – representations of fluid parameters as functions of spatial coordinate

- the velocity field

$$\vec{V} = u(x, y, z, t)\vec{i} + v(x, y, z, t)\vec{j} + w(x, y, z, t)\vec{k}$$



$$\frac{d\vec{r}_A}{dt} = \vec{V}_A$$

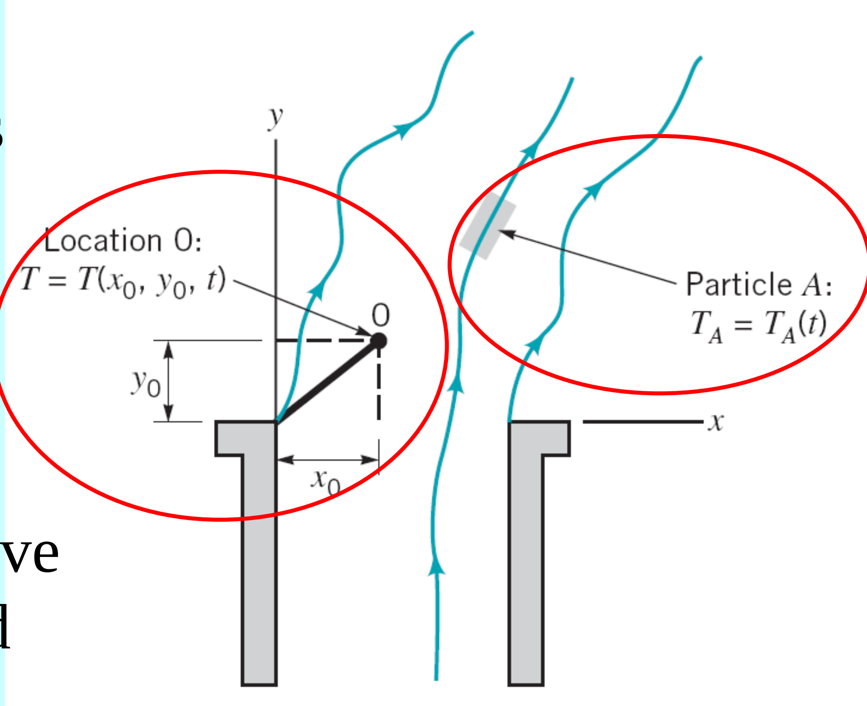
$$\vec{V} = \vec{V}(x, y, z, t)$$

$$V = \left|\vec{V}\right| = \left(u^2 + v^2 + w^2\right)^{1/2}$$

A change in velocity results in an acceleration which may be due to a change in speed and/or direction.

4.1.1 Eulerian and Lagrangian Flow Descriptions

- **Eulerian method**: the fluid motion is given by completely prescribing the necessary properties as functions of space and time.
- From this method, we obtain information about the flow in terms of what happens at fixed points in space as the fluid flows past those points.

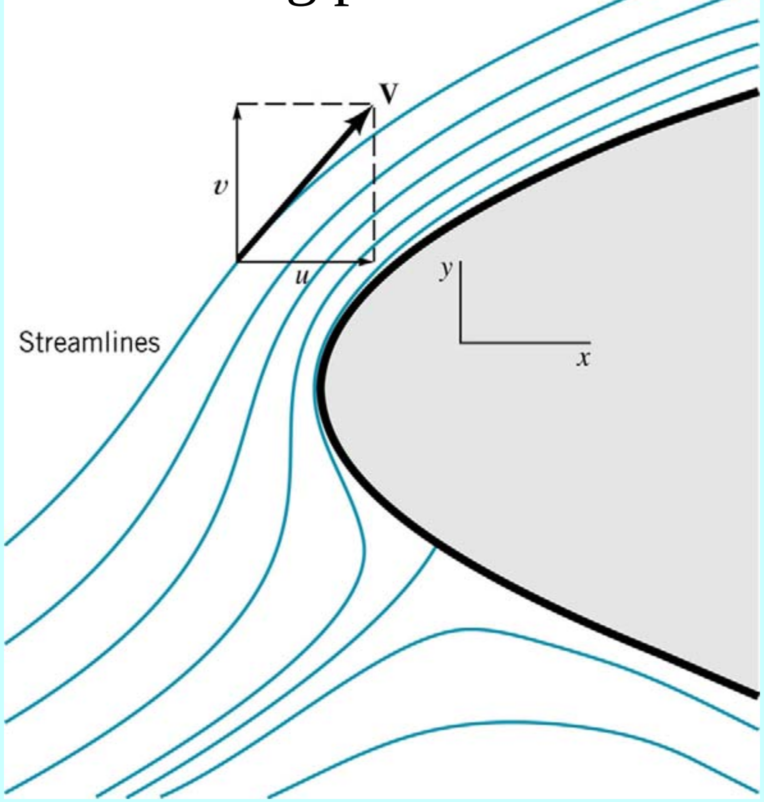


- **Lagrangian method**: following individual fluid particles as they move about and determining how the fluid properties associated with these particles change as a function of time.

V4.3 Cylinder-velocity vectors
V4.4 Follow the particles
V4.5 Follow the particles

4.1.4 Streamlines, Streaklines and Pathlines

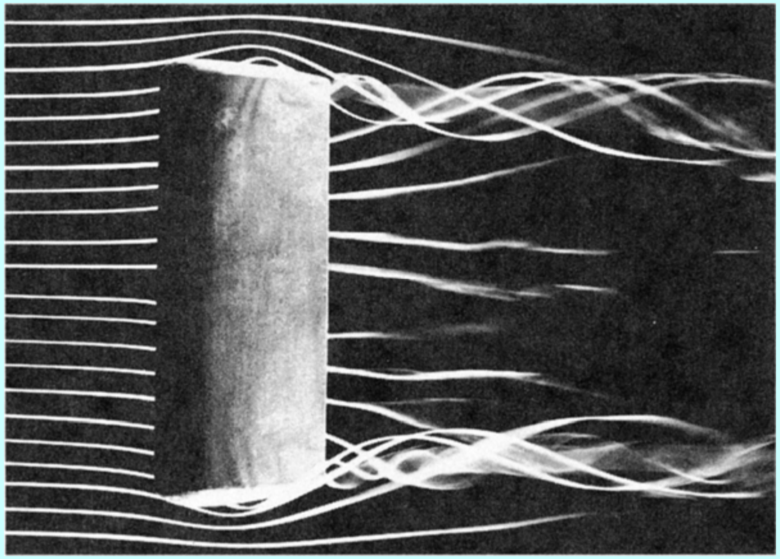
- A **streamline** is a line that is everywhere tangent to the velocity field.
- A streakline consists of all particles in a flow that have previously passed through a common point.
- A pathline is a line traced out by a given flowing particle.



V4.9 streamlines
V4.10 streaklines
V4.1 streaklines

4.1.4 Streamlines, Streaklines and Pathlines

- For steady flows, streamlines, streaklines and pathlines all coincide. This is not true for unsteady flows.



- Unsteady streamlines are difficult to generate experimentally, but easy to draw in numerical computation.
- On the contrary, streaklines are more of a lab tool than an analytical tool.
- How can you determine the unsteady pathline of a moving particle?

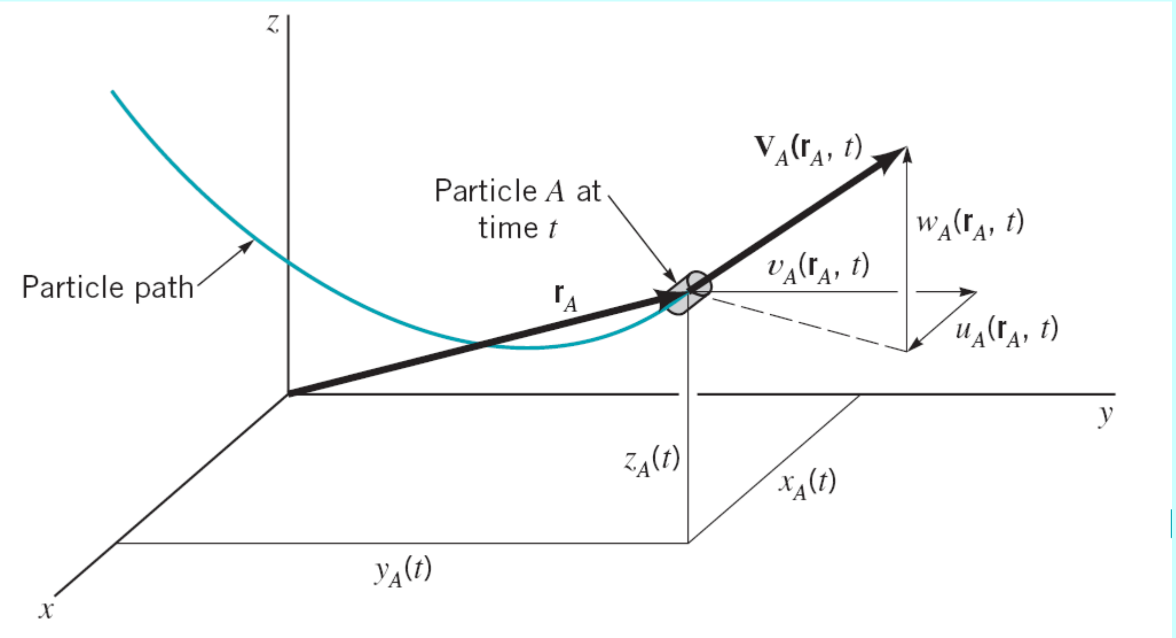
4.2 The Acceleration Field

- The acceleration of a particle is the time rate change of its velocity.
- For unsteady flows the velocity at a given point in space may vary with time, giving rise to a portion of the fluid acceleration.
- In addition, a fluid particle may experience an acceleration because its velocity changes as it flows from one point to another in space.

4.2.1 The Material Derivative

- Consider a particle moving along its pathline

$$\overrightarrow{V_A} = \overrightarrow{V_A}\left(\overrightarrow{r_A}, t\right) = \overrightarrow{V_A}\left[x_A(t), y_A(t), z_A(t), t\right]$$



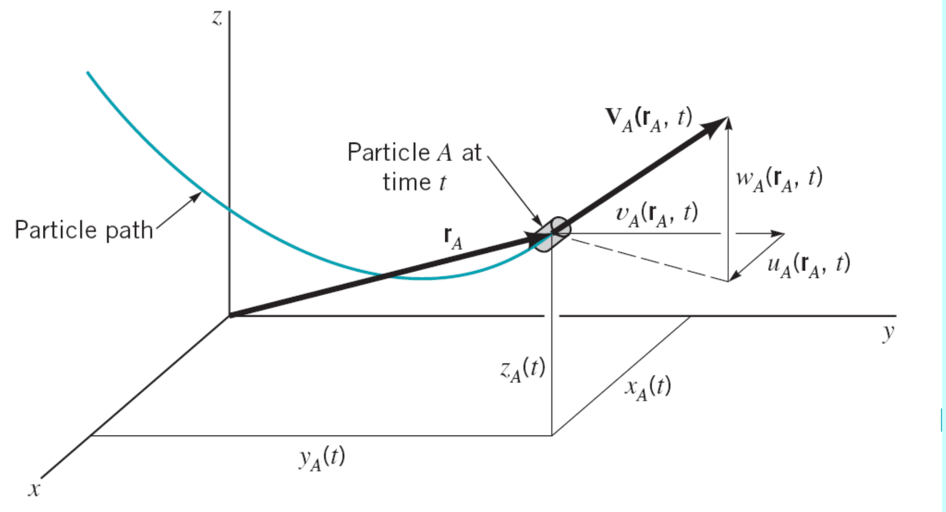
The Material Derivative

- Thus the acceleration of particle A,

$$\overrightarrow{V_A} = \overrightarrow{V_A}\left(\overrightarrow{r_A}, t\right) = \overrightarrow{V_A}\left[x_A(t), y_A(t), z_A(t), t\right]$$

$$\overrightarrow{a_A}(t) = \frac{d\overrightarrow{V_A}}{dt} = \frac{\partial \overrightarrow{V_A}}{\partial t} + \frac{\partial \overrightarrow{V_A}}{\partial x}\frac{dx_A}{dt} + \frac{\partial \overrightarrow{V_A}}{\partial y}\frac{dy_A}{dt} + \frac{\partial \overrightarrow{V_A}}{\partial z}\frac{dz_A}{dt}$$

$$= \frac{\partial \overrightarrow{V_A}}{\partial t} + u_A\frac{\partial \overrightarrow{V_A}}{\partial x} + v_A\frac{\partial \overrightarrow{V_A}}{\partial y} + w_A\frac{\partial \overrightarrow{V_A}}{\partial z}$$



Acceleration

- This is valid for any particle

$$\vec{a} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}$$

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

Material derivative

- Acceleration:

Associated with time variation

$$\vec{a} = \frac{D\vec{V}}{Dt}, \quad \frac{D\vec{V}}{Dt} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}$$

Associated with space variation

- Total derivative, material derivative or substantial derivative

$$\frac{D(\ )}{Dt} = \frac{\partial(\ )}{\partial t} + u\frac{\partial(\ )}{\partial x} + v\frac{\partial(\ )}{\partial y} + w\frac{\partial(\ )}{\partial z}$$

$$= \frac{\partial(\ )}{\partial t} + (\vec{V}\cdot\nabla)(\ )$$

Material derivative

- The material derivative of any variable is the rate at which that variable changes with time for a given particle (as seen by one moving along with the fluid – the **Lagrangian descriptions**)
- If velocity is known, the time rate change of temperature can be expressed as,

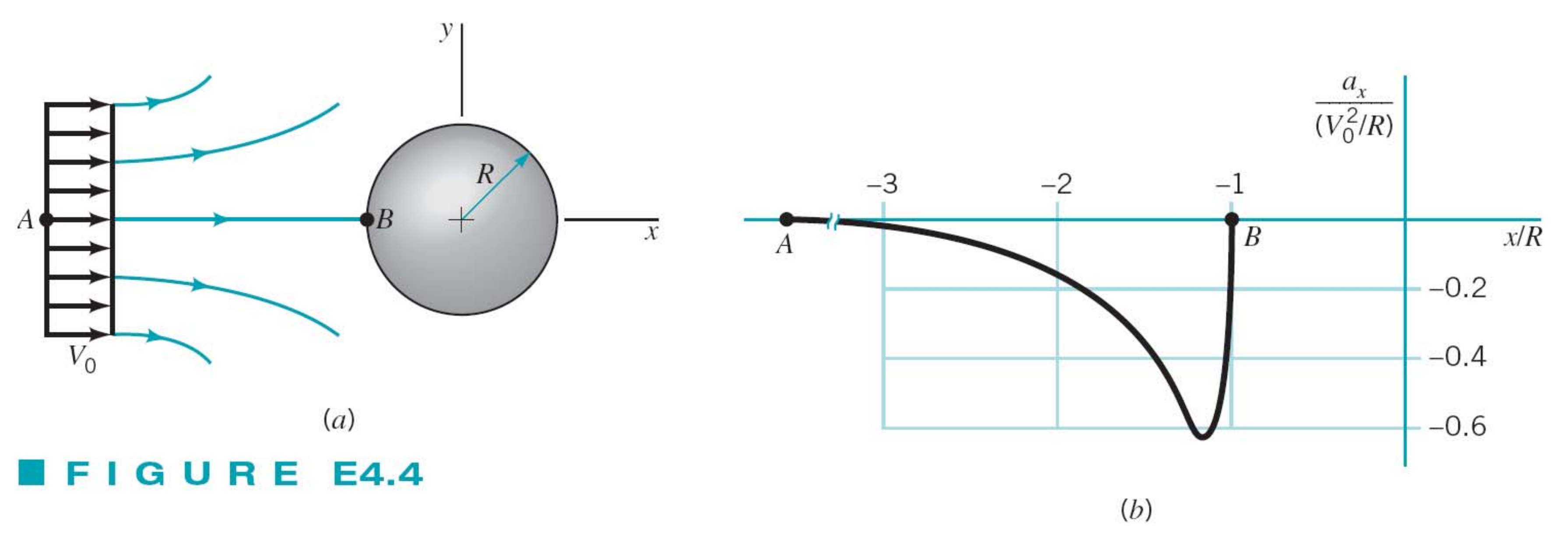
$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}$$

$$= \frac{\partial T}{\partial t} + (\vec{V} \cdot \nabla)T$$

Example: the temperature of a passenger experienced on a train starting from Taipei on 9am and arriving at Kaohsiung on 12.

Acceleration along a streamline

$$\vec{V} = u(x)\ \vec{i} = V_0\left(1+\frac{R^3}{x^3}\right)\vec{i}, \qquad \vec{a} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} = \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right)\vec{i}$$



■ FIGURE E4.4

$$\vec{a} = V_o\left(1+\frac{R^3}{x^3}\right)V_o\left[R^3(-3x^{-4})\right]\vec{i}$$

4.2.2 Unsteady Effects

For steady flow $\partial(\ )/\partial t \equiv 0,$ there is no change in flow parameters at a fixed point in space.

For unsteady flow $\partial(\ )/\partial t \neq 0.$

$\downarrow$ spatial (convective) derivative

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \vec{V} \cdot \nabla T \quad \text{(for an unstirred cup of coffee} \frac{DT}{Dt} \to \frac{\partial T}{\partial t} < 0\text{)}$$

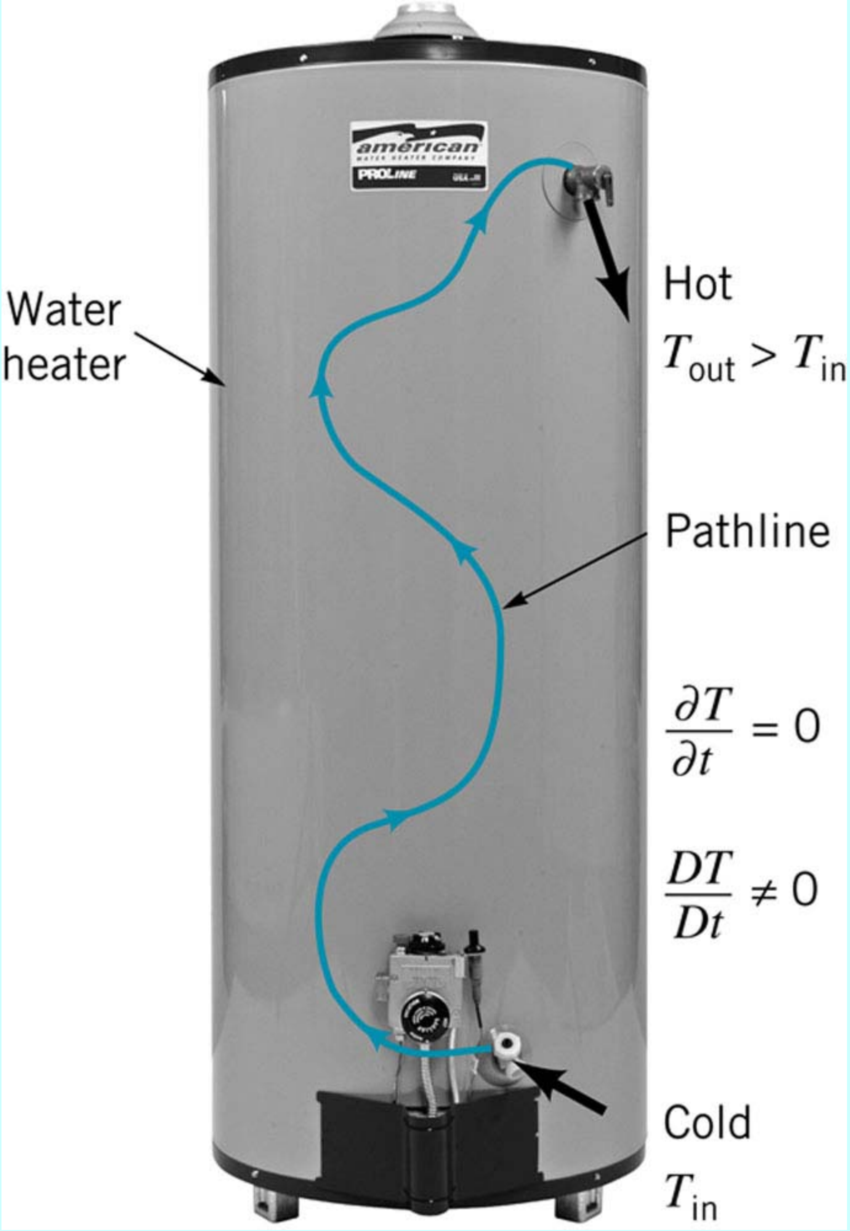
$\uparrow$ time (local) derivative

$$\frac{D\vec{V}}{Dt} = \frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \nabla \vec{V}$$

$\uparrow$ local acceleration

V4.12 Unsteady flow

4.2.3 Convective Effects



$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \vec{V} \cdot \nabla T$$

$$\frac{DT}{Dt} = 0 + u_s \frac{\partial T}{\partial s}$$

$$= 0 + u_s \frac{T_{out} - T_{in}}{\Delta s}$$

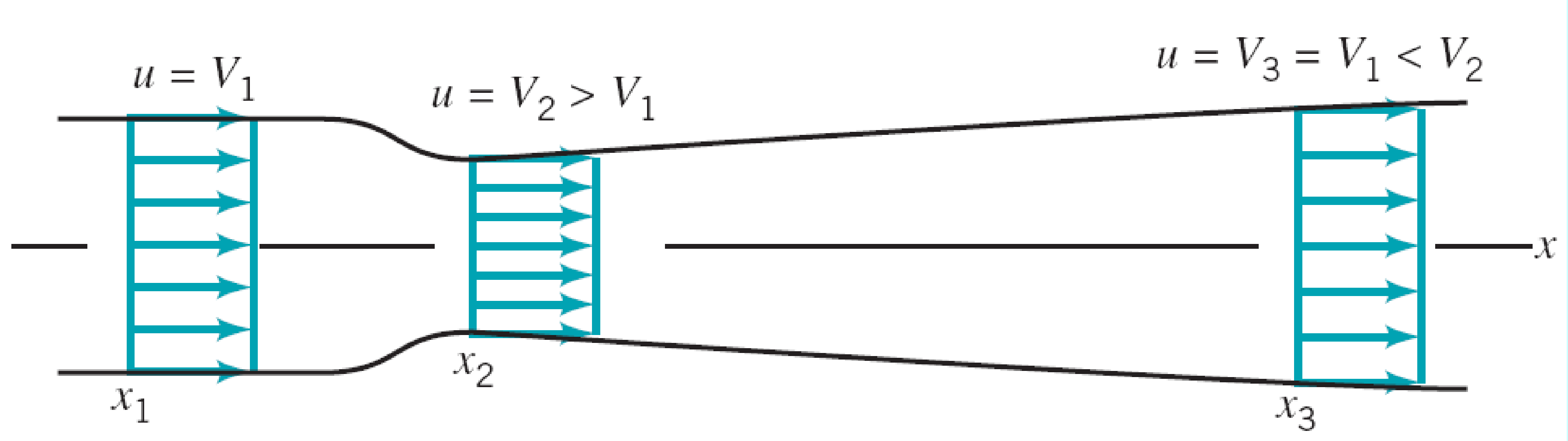
4.2.3 Convective Effects

$\downarrow$ convective acceleration

$$\frac{D\vec{V}}{Dt} = \frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \nabla \vec{V}$$

$\uparrow$ local acceleration

$$\frac{Du}{Dt} = 0 + u\frac{\partial u}{\partial x}$$



4.2.4 Streamline Coordinates

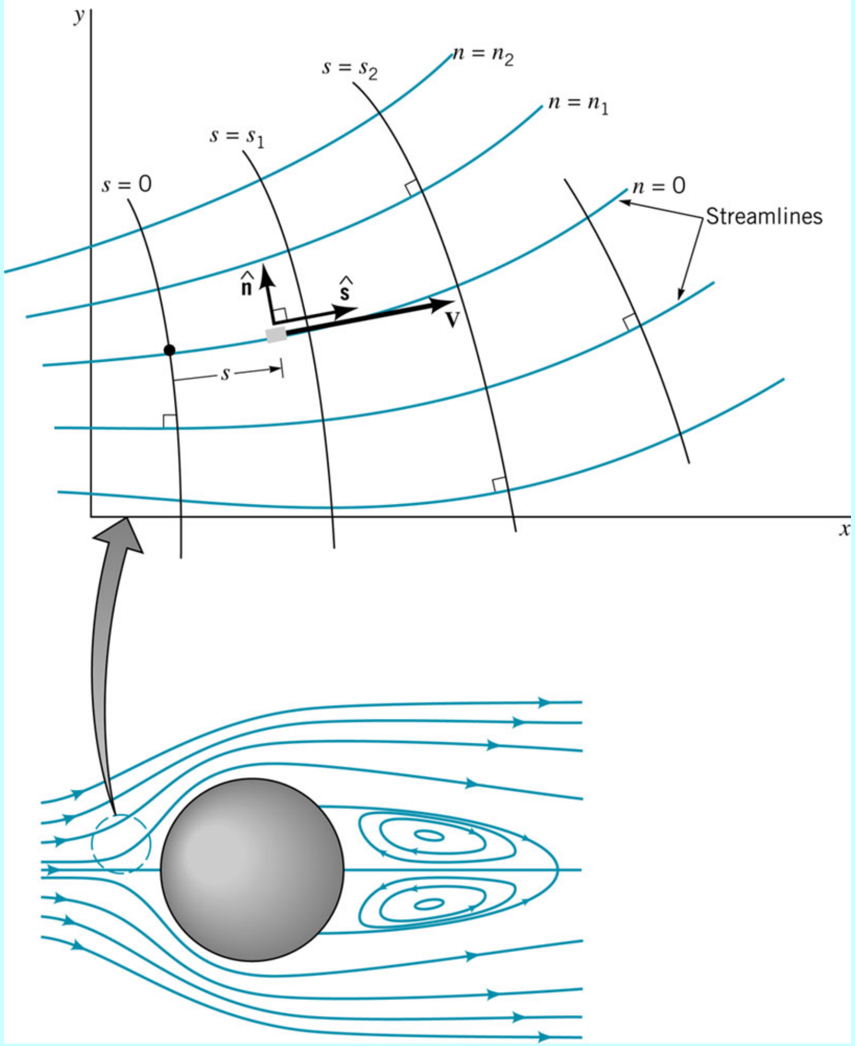
- In many flow situations it is convenient to use a coordinate system defined in terms of the streamlines of the flow.

$$\vec{V} = V\vec{s}$$

$$\vec{a} = \frac{D\vec{V}}{Dt} = \frac{DV}{Dt}\vec{s} + V\frac{D\vec{s}}{Dt}$$

$$= \left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial s}\frac{ds}{dt} + \frac{\partial V}{\partial n}\frac{dn}{dt}\right)\vec{s}$$

$$+V\left(\frac{\partial \vec{s}}{\partial t} + \frac{\partial \vec{s}}{\partial s}\frac{ds}{dt} + \frac{\partial \vec{s}}{\partial n}\frac{dn}{dt}\right)$$



V4.13 Streamline coordinates

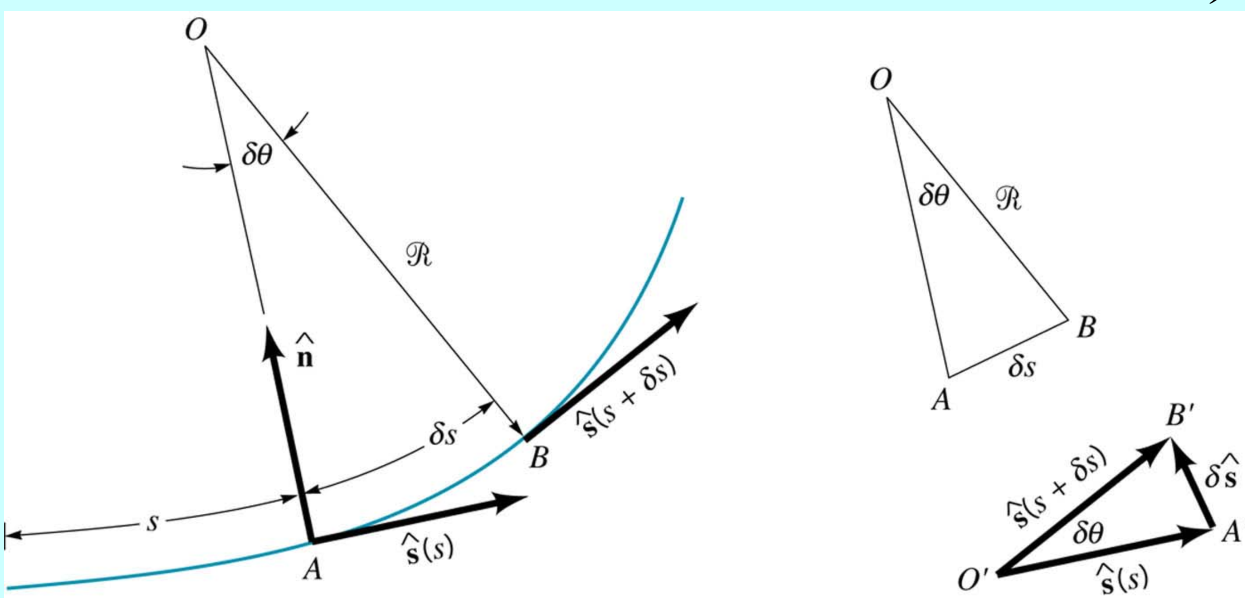
4.2.4 Streamline Coordinates

Steady flow

$$\vec{a} = \left(V\frac{\partial V}{\partial s}\right)\vec{s} + V\left(V\frac{\partial \vec{s}}{\partial s}\right)$$

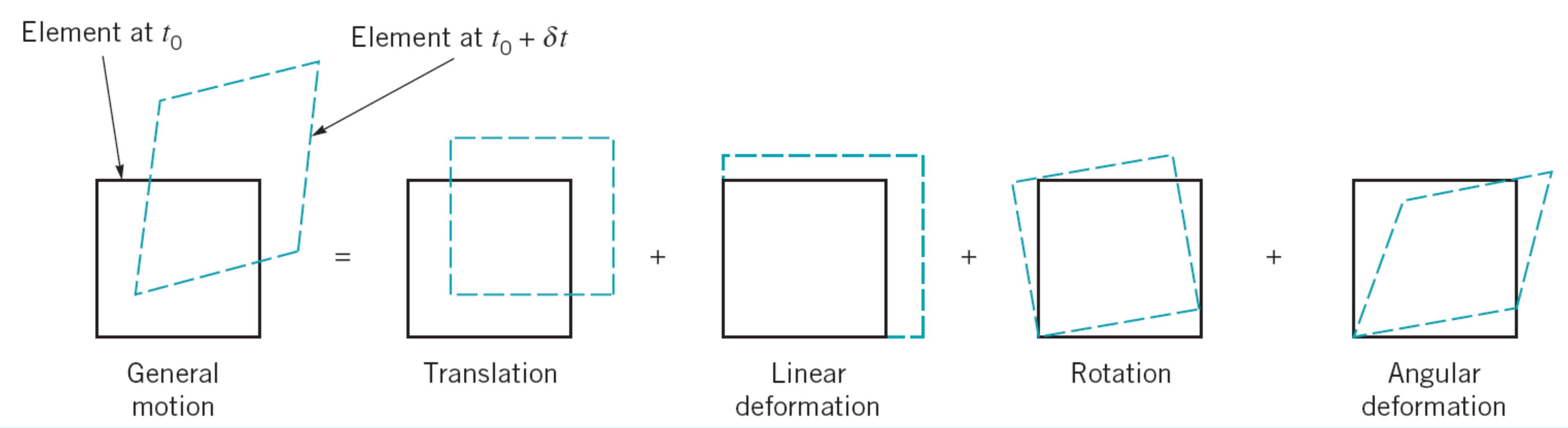
$$= V\frac{\partial V}{\partial s}\vec{s} + \frac{V^2}{R}\vec{n} \qquad \text{or} \quad a_s = V\frac{\partial V}{\partial s},\ \ a_n = \frac{V^2}{R}$$

$$\left(\because \frac{\delta s}{R} = \frac{\left|\delta\vec{s}\right|}{\left|\vec{s}\right|} = \left|\delta\vec{s}\right|,\ \text{ or }\ \left|\frac{\delta\vec{s}}{\delta s}\right| = \frac{1}{R}, \frac{\partial\vec{s}}{\partial s} = \lim_{\delta s\to 0}\frac{\delta\vec{s}}{\delta s} = \frac{\vec{n}}{R}\right)$$



6.1 Fluid Element Kinematics

- Types of motion and deformation for a fluid element.



6.1.1 Velocity and Acceleration Fields Revisited

- Velocity field representation

$$\vec{V} = \vec{V}(x, y, z, t) \quad \text{or} \quad \vec{V} = u\vec{i} + v\vec{j} + w\vec{k}$$

- Acceleration of a particle

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$\vec{a} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}$$

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

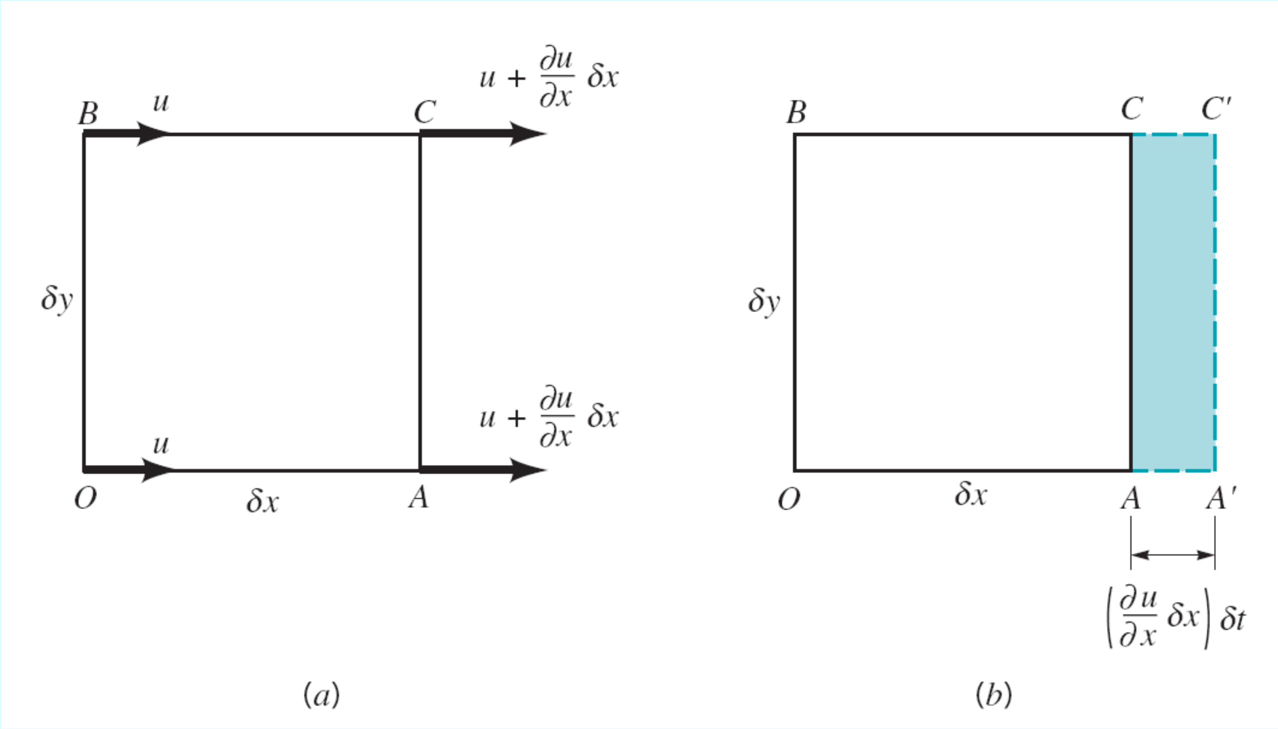
$$\vec{a} = \frac{D\vec{V}}{Dt} = \frac{\partial \vec{V}}{\partial t} + \left(\vec{V} \cdot \nabla\right)\vec{V}$$

$$\nabla(\ ) = \frac{\partial(\ )}{\partial x}\vec{i} + \frac{\partial(\ )}{\partial y}\vec{j} + \frac{\partial(\ )}{\partial z}\vec{k}$$

6.1.2 Linear Motion and Deformation

- variations of the velocity in the direction of velocity, $\frac{\partial u}{\partial x}$, $\frac{\partial v}{\partial y}$, $\frac{\partial w}{\partial z}$ cause a linear *stretching* deformation.

  Consider the *x*-component deformation:



Linear Motion and Deformation

The change in the original volume, $\delta \mathcal{V} = \delta x \delta y \delta z$, due to $\partial u / \partial x$:

$$\text{Change in } \delta \mathcal{V} = \left(\frac{\partial u}{\partial x}\delta x\right)(\delta y \delta z)(\delta t)$$

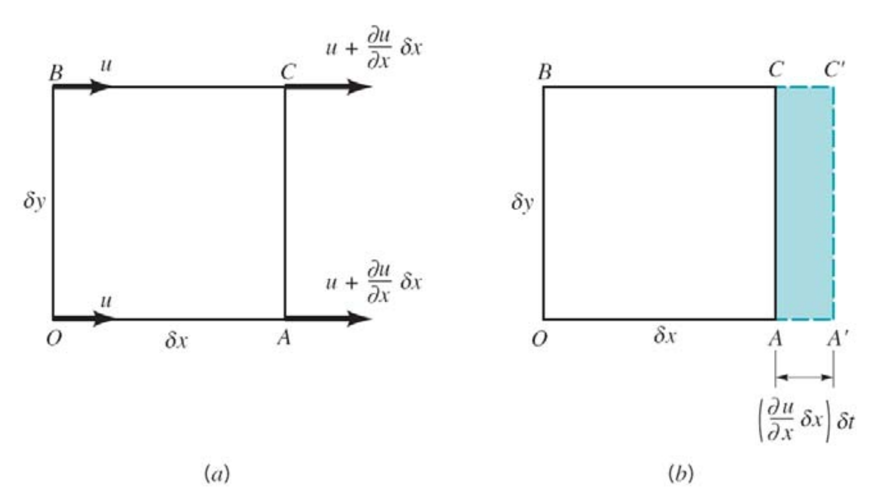
Rate change of $\delta \mathcal{V}$ per unit volume due to $\partial u / \partial x$:

$$\frac{1}{\delta \mathcal{V}}\frac{d(\delta \mathcal{V})}{dt} = \lim_{\delta t \to 0}\left[\frac{\left(\partial u / \partial x\right)\delta t}{\delta t}\right] = \frac{\partial u}{\partial x}$$

If velocity gradient $\partial v / \partial y$ and $\partial w / \partial z$ are also present, then

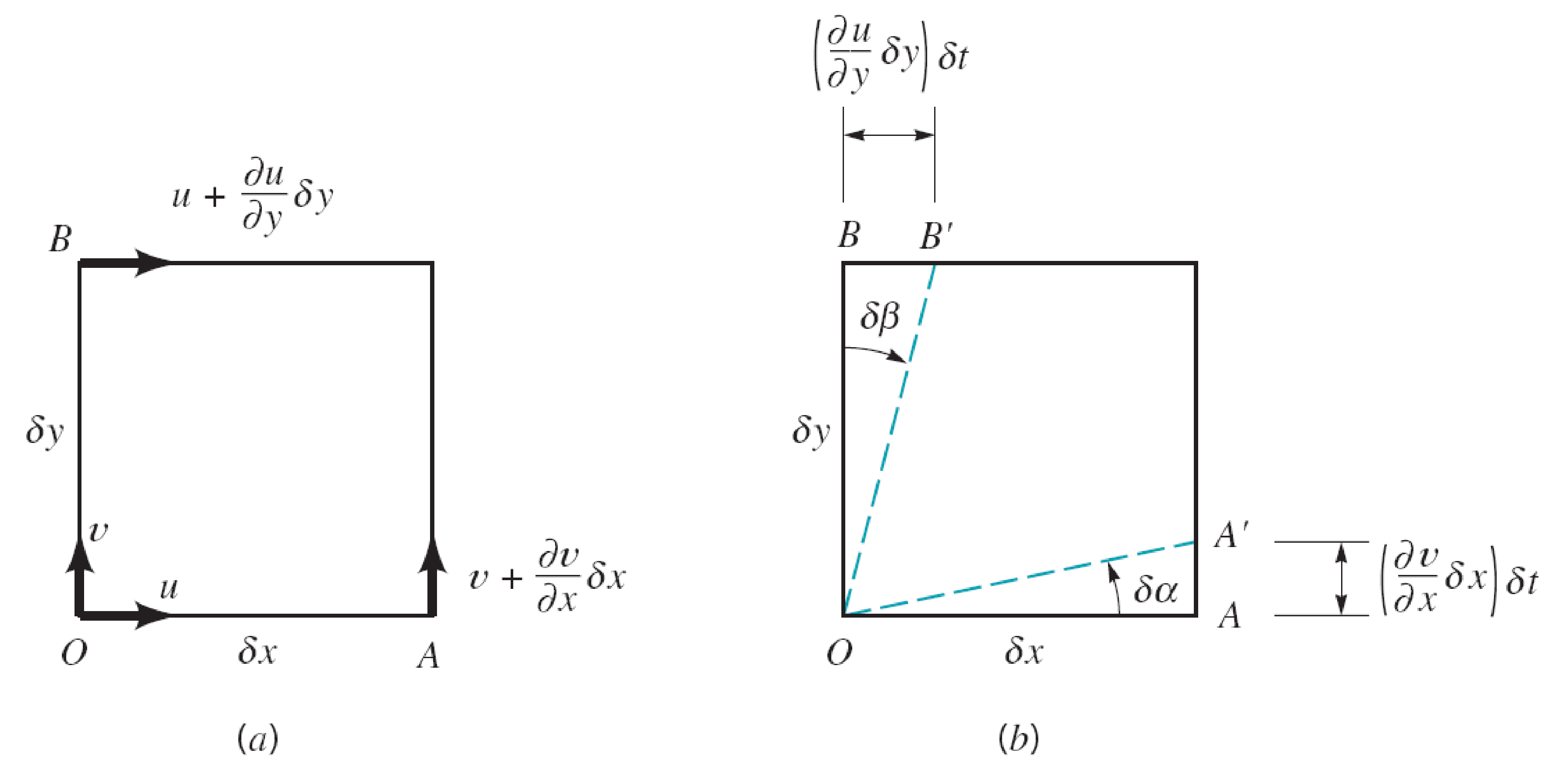
$$\frac{1}{\delta \mathcal{V}}\frac{d(\delta \mathcal{V})}{dt} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla \cdot \vec{V} \quad \leftarrow \quad \textbf{volumetric dilatation rate}$$

- The volume of a fluid may change as the element moves from one location to another in the flow field.
- For incompressible fluid, the volumetric dilation rate is zero.



6.1.3 Angular Motion and Deformation

- Consider an element under rotation and angular deformation



V6.3 Shear deformation

Angular Motion and Deformation

- the angular velocity of OA is

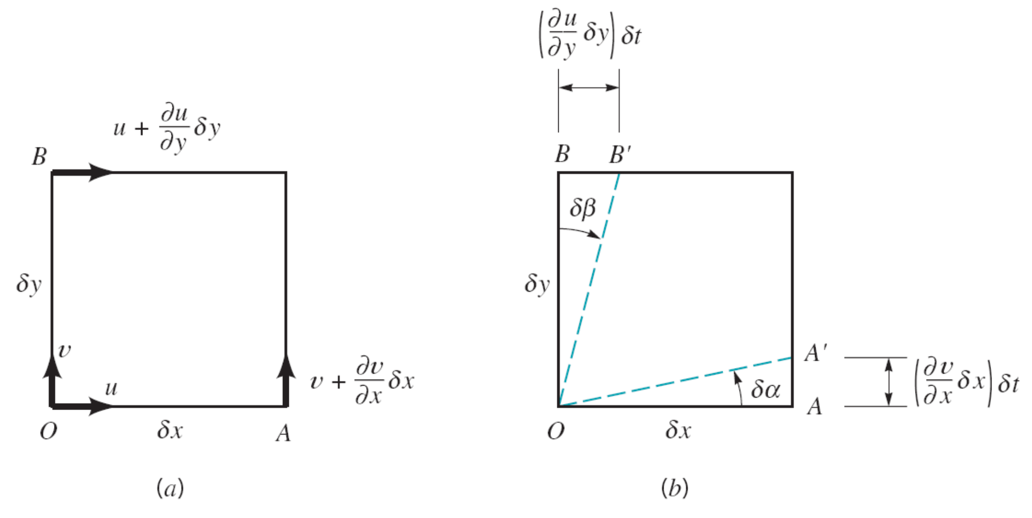
$$\omega_{OA} = \lim_{\delta t \to 0} \frac{\delta \alpha}{\delta t}$$

- For small angles

$$\tan \delta\alpha \approx \delta\alpha = \frac{\frac{\partial v}{\partial x}\delta x \delta t}{\delta x} = \frac{\partial v}{\partial x}\delta t$$

so that

$$\omega_{OA} = \lim_{\delta t \to 0}\left[\frac{(\delta v / \delta x)\delta t}{\delta t}\right] = \frac{\delta v}{\delta x}$$



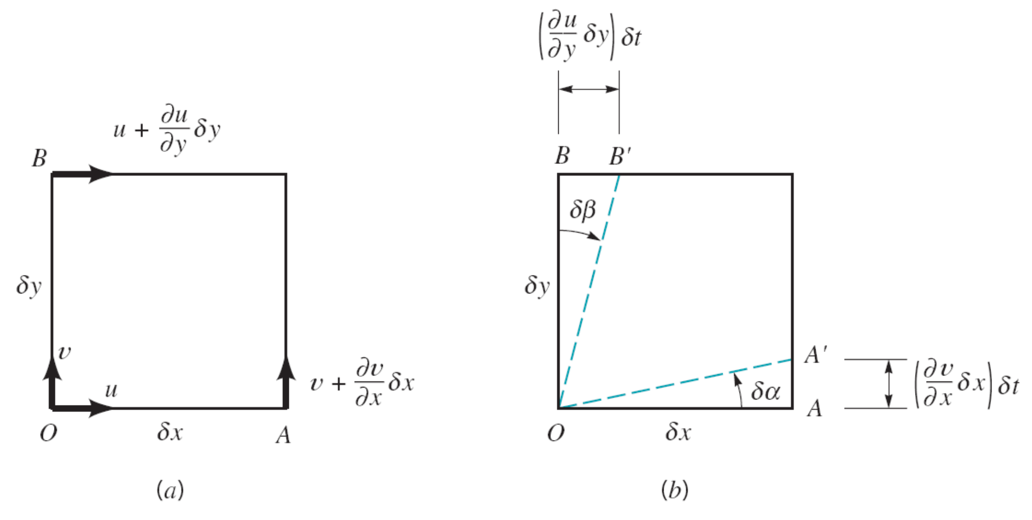
(if $\frac{\partial v}{\partial x}$ is positive then ω_{OA} will be counterclockwise)

Angular Motion and Deformation

- the angular velocity of the line OB is

$$\omega_{OB} = \lim_{\delta t \to 0} \frac{\delta\beta}{\delta t}$$

$$\tan \delta\beta \approx \delta\beta = \frac{\frac{\partial u}{\partial y}\delta y \delta t}{\delta y} = \frac{\partial u}{\partial y}\delta t$$



so that

$$\omega_{OB} = \lim_{\delta t \to 0}\left[\frac{(\partial u / \partial y)\delta t}{\delta t}\right] = \frac{\partial u}{\partial y}$$

(if $\frac{\partial u}{\partial y}$ is positive, ω_{OB} will be clockwise)

Angular rotation

V4.6 Flow past a wing

- The **rotation**, ω_z , of the element about the z axis is defined as the average of the angular velocities ω_{OA} and ω_{OB} , if counterclockwise is considered to be positive, then,

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

similarly

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right), \qquad \omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

thus

$$\vec{\omega} = \omega_x \vec{i} + \omega_y \vec{j} + \omega_x \vec{k} = \frac{1}{2} curl\, \vec{V} = \frac{1}{2} \nabla \times \vec{V}$$

$$\frac{1}{2}\nabla \times \vec{V} = \frac{1}{2}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}$$

Definition of vorticity

- Define **vorticity** ξ

  $$\vec{\xi} = 2\vec{\omega} = \nabla \times \vec{V}$$

  The fluid element will rotate about *z* axis as an *undeformed* block (ie: $\omega_{OA} = -\omega_{OB}$)

  only when $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

  Otherwise the rotation will be associated with an *angular deformation*.

- If $\dfrac{\partial u}{\partial y} = \dfrac{\partial v}{\partial x}$ or $\nabla \times \vec{V} = 0$, then the rotation (and the vorticity) are zero, and flow fields are termed **irrotational**.

Different types of angular motions

- Solid body rotation

$$u_\phi = \Omega r \qquad u_r = u_z = 0$$

$$\omega_z = 2\Omega \qquad \omega_r = \omega_\phi = 0$$

$$\omega_z = \frac{1}{r}\frac{\partial}{\partial r}\left(r u_\phi\right) - \frac{1}{r}\frac{\partial u_r}{\partial \theta}$$

- Free vortex

$$u_\phi = \frac{k}{r} \qquad u_r = u_z = 0$$

$$\omega_\phi = 0 \qquad \omega_r = 0$$

$$\omega_z = \frac{1}{r}\frac{\partial}{\partial r}\left(r u_\phi\right) = 0 \quad \text{for} \quad r \neq 0$$

Angular Deformation

- Apart form rotation associated with these derivatives $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial x}$, these derivatives can cause the element to undergo an angular deformation, which results in a change in shape of the element.

- The change in the original right angle is termed the shearing strain $\delta\gamma$,

$$\delta\gamma = \delta\alpha + \delta\beta$$

where $\delta\gamma$ is considered to be positive if the original right angle is decreasing.

Angular Deformation

- Rate of shearing strain or rate of angular deformation

$$\dot{\gamma} = \lim_{\delta t \to 0} \frac{\delta \gamma}{\delta t} = \lim_{\delta t \to 0} \left[\frac{\frac{\partial v}{\partial x}\delta t + \frac{\partial u}{\partial y}\delta t}{\delta t} \right]$$

$$= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

The rate of angular deformation is related to a corresponding shearing stress which causes the fluid element to change in shape.

If $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, the rate of angular deformation is zero and this condition indicates that the element is simply rotating as an *undeformed* block.

6.2 Conservation of Mass

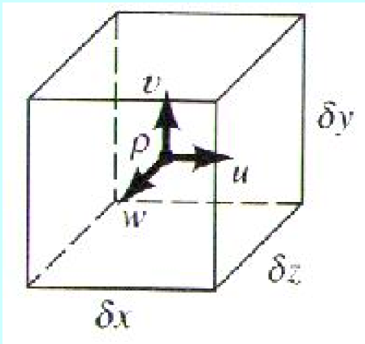
Conservation of mass: $$\frac{DM_{sys}}{Dt} = 0$$

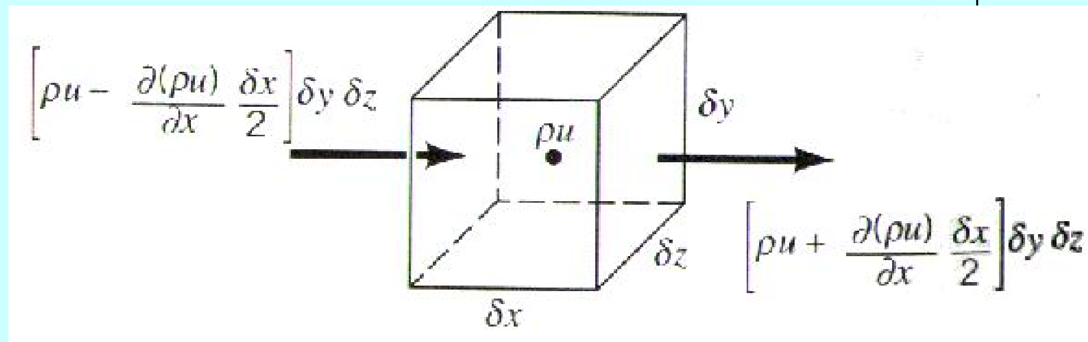
In control volume representation (**continuity equation**):

$$\frac{\partial}{\partial t}\int_{cv} \rho d\forall + \int_{cs} \rho \vec{V} \bullet \vec{n} dA = 0 \qquad (6.19)$$

To obtain the differential form of the continuity equation, Eq. 6.19 is applied to an infinitesimal control volume.

6.2.1 Differential Form of Continuity Equation





$$\frac{\partial}{\partial t}\int \rho d\mathcal{V} \equiv \frac{\partial \rho}{\partial t}\delta x \delta y \delta z$$

Net mass flow in the direction $\left[\rho u + \frac{\partial \rho u}{\partial x}\frac{\delta x}{2}\right]\delta y \delta z - \left[\rho u - \frac{\partial \rho u}{\partial x}\frac{\delta x}{2}\right]\delta y \delta z = \frac{\partial \rho u}{\partial x}\delta x \delta y \delta z$

Net mass flow in the *y* direction $\frac{\partial \rho v}{\rho y}\delta x \delta y \delta z$

Net mass flow in the *z* direction $\frac{\partial \rho w}{\rho z}\delta x \delta y \delta z$

Net rate of mass out of flow $\left[\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z}\right]\delta x \delta y \delta z$

Differential Form of Continuity Equation

- Thus conservation of mass become

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0 \qquad \textbf{(continuity equation)}$$

- In vector form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{V} = 0$$

- For steady compressible flow

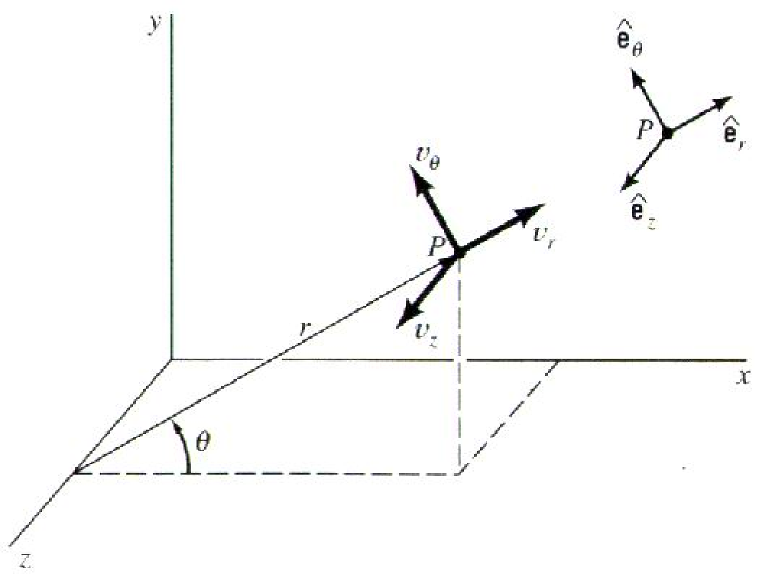
$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0 \qquad \nabla \cdot \rho \vec{V} = 0$$

- For incompressible flow

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad \nabla \cdot \vec{V} = 0$$

6.2.2 Cylindrical Polar Coordinates

- The differential form of continuity equation



$$\frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial (r\rho v_r)}{\partial r}+\frac{1}{r}\frac{\partial (\rho v_\theta)}{\partial \theta}+\frac{\partial \rho v_z}{\partial z}=0$$

6.2.3 The Stream Function

- For 2-D incompressible plane flow then,

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

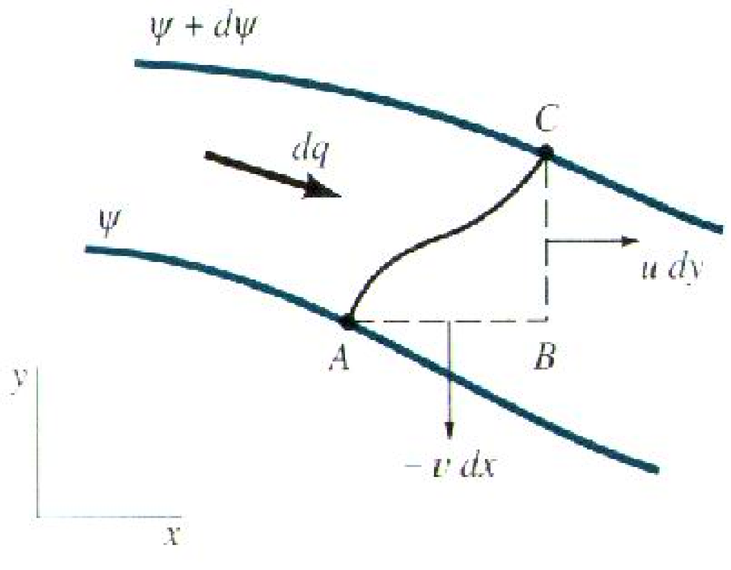
- Define a **stream function** $\psi(x,y)$ such that

$$u=\frac{\partial \psi}{\partial y} \qquad v=-\frac{\partial \psi}{\partial x}$$

then
$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right)+\frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial y}\right)=\frac{\partial^2 \psi}{\partial x \partial y}-\frac{\partial^2 \psi}{\partial y \partial x}=0$$

- For velocity expressed in forms of the stream function, the conservation of mass will be satisfied.

The Stream Function



- Lines along constant ψ are stream lines

  Definition of stream line $\frac{dy}{dx} = \frac{v}{u}$

  Thus change of ψ, from (x, y) to $(x + dx, y + dy)$

  $$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = -vdx + udy$$

  Along a line of constant ψ of we have $d\psi = 0$

  $$-vdx + udy = 0 \qquad \frac{dy}{dx} = \frac{v}{u}$$

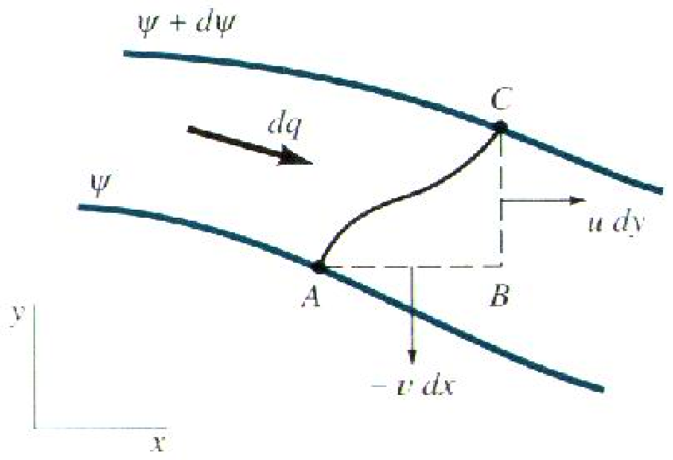
  which is the defining equation for a streamline.

- Thus we can use ψ to plot streamline.
- The actual numerical value of a stream line is not important but the change in the value of ψ is related to the volume flow rate.

The Stream Function

Note：Flow never crosses streamline, since by definition the velocity is tangent to the streamlines.

- Volume rate of flow (per unit width perpendicular to the *x-y*



$$dq = udy - vdx$$

$$= \frac{\partial \psi}{\partial y} dy + \frac{\partial \psi}{\partial x} dx = d\psi$$

$$q = \int_{\psi_1}^{\psi_2} d\psi = \psi_2 - \psi_1$$

If $\psi_2 > \psi_1$ then q is positive and vice versa.

- In cylindrical coordinates the incompressible continuity equation becomes, $$\frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} = 0$$

Then, $v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$ $\qquad v_\theta = -\frac{\partial \psi}{\partial r}$

Ex 6.3 Stream function

6.3 Conservation of Linear Momentum

- Linear momentum equation

$$\vec{F} = \frac{D}{Dt}\int_{sys} \vec{V}\,dm$$

or

$$\sum \vec{F}_{\substack{\text{contents of the}\\ \text{control volume}}} = \frac{\partial}{\partial t}\int_{CV} \vec{V}\rho\, d\forall + \int_{CS} \vec{V}\rho\vec{V}\cdot\vec{n}\,dA$$

- Consider a differential system with δm and $\delta\forall$

then

$$\delta\vec{F} = \frac{D\left(\vec{V}\delta m\right)}{Dt}$$

- Using the system approach then

$$\delta\vec{F} = \delta m\frac{D\vec{V}}{Dt} = \delta m\vec{a}$$

6.3.1 Descriptions of Force Acting on the Differential Element

- Two types of forces need to be considered

  surface forces：which action the surfaces of the differential element.

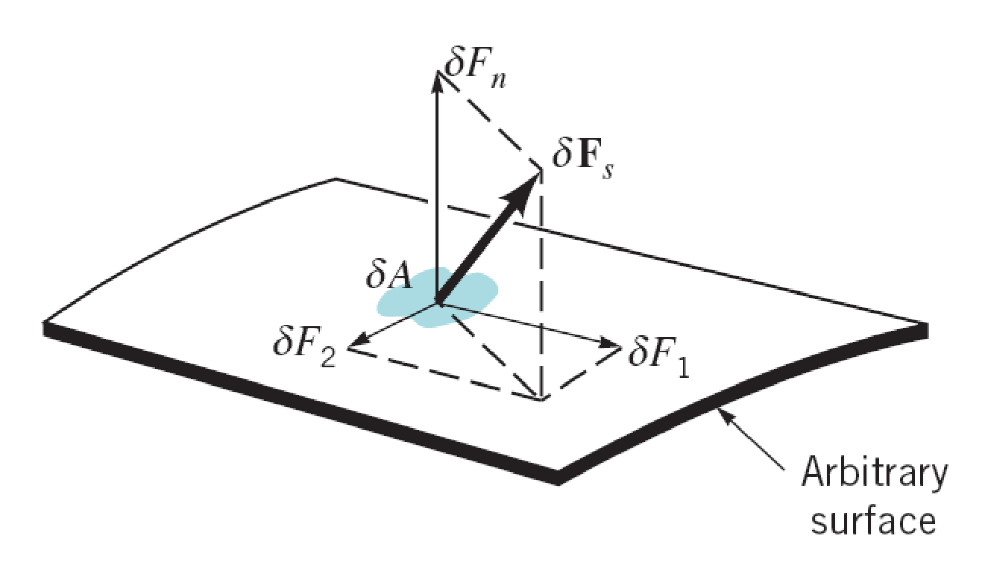
  body forces：which are distributed throughout the element.

- For simplicity, the only body force considered is the weight of the element,

$$\delta \vec{F}_b = \delta m \vec{g}$$

or $\delta F_{bx} = \delta m g_x \qquad \delta F_{by} = \delta m g_y \qquad \delta F_{bz} = \delta m g_z$

- Surface force act on the element as a result of its interaction with its surroundings (the components depend on the area orientation)



Where δF_n is normal to the area δA and δF_1 and δF_2 are parallel to the area and orthogonal to each other.

- The normal stress σ_n is defined as,

$$\sigma_n = \lim_{\delta A \to 0} \frac{\delta F_n}{\delta A}$$

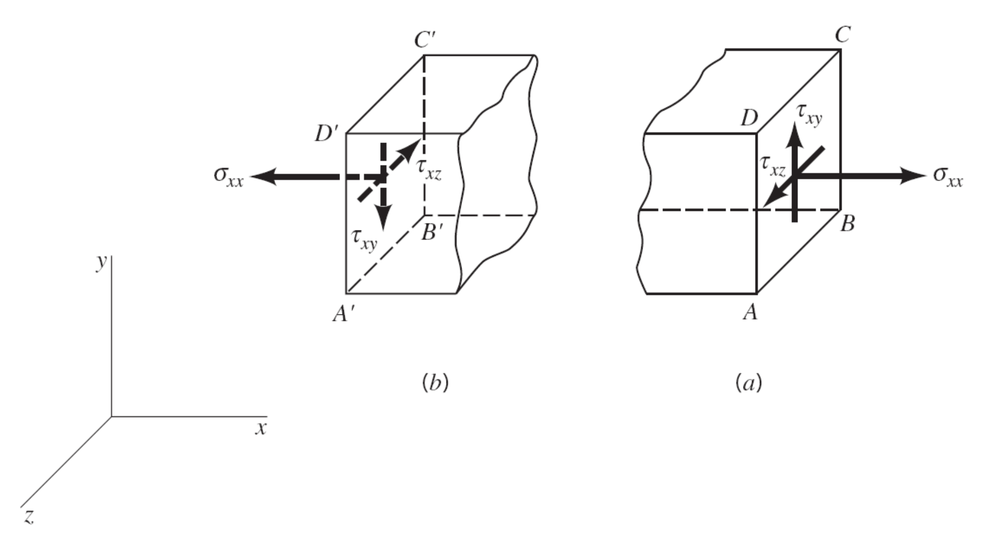
and the shearing stresses are define as

$$\tau_1 = \lim_{\delta A \to 0} \frac{\delta F_1}{\delta A} \qquad \tau_2 = \lim_{\delta A \to 0} \frac{\delta F_2}{\delta A}$$

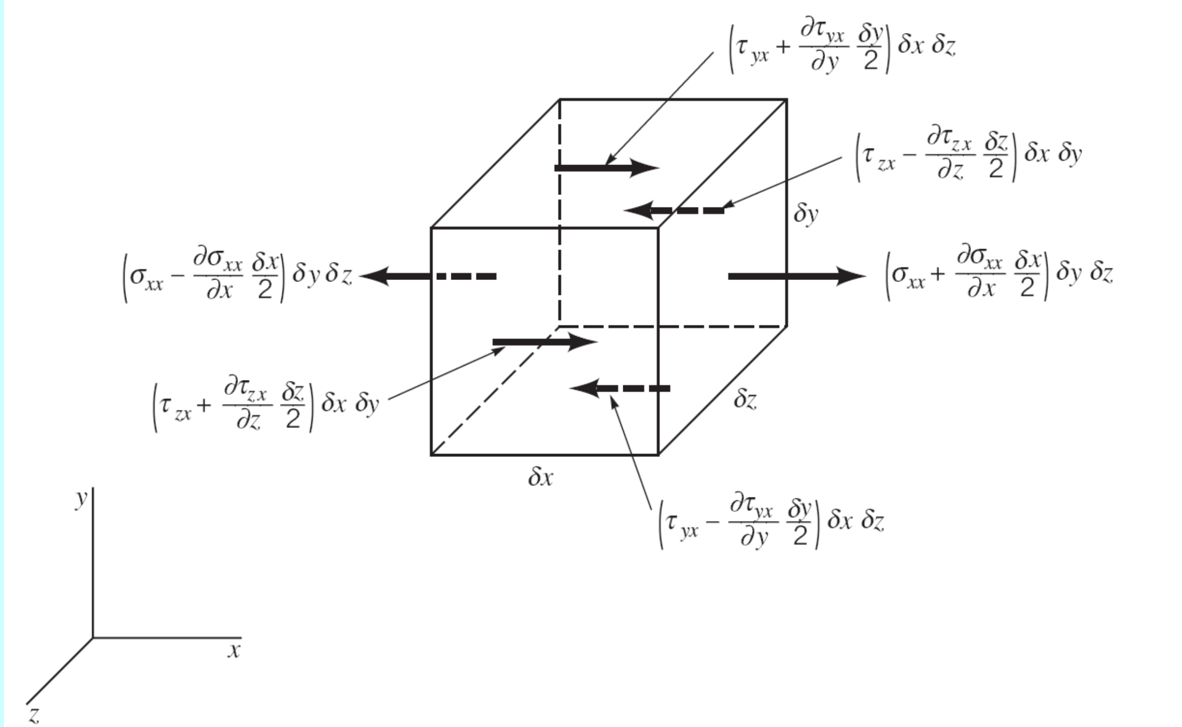
we use σ for normal stresses and τ for shear stresses.

- Sign of stresses

Positive sign for the stress as positive coordinate direction on the surfaces for which the outward normal is in the positive coordinate direction.



Note：Positive normal stresses are tensile stresses, ie, they tend to stretch the material.



Thus

$$\delta F_{sx} = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)\delta x \delta y \delta z$$

$$\delta F_{sy} = \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right)\delta x \delta y \delta z$$

$$\delta F_{sz} = \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}\right)\delta x \delta y \delta z$$

$$\delta \vec{F}_s = \delta F_{sx}\vec{i} + \delta F_{sy}\vec{j} + \delta F_{sz}\vec{k}$$

$$\delta \vec{F} = \delta \vec{F}_s + \delta \vec{F}_b$$

6.3.2 Equation of Motion

$$\delta \vec{F}_x = \delta m\, a_x, \quad \delta \vec{F}_y = \delta m\, a_y, \quad \delta \vec{F}_z = \delta m\, a_z$$

$$\delta m = \rho \delta x \, \delta y \, \delta z$$

Thus

$$\begin{aligned}
\rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} &= \rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) \\
\rho g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} &= \rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) \qquad (6.50) \\
\rho g_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} &= \rho \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right)
\end{aligned}$$

PART B

Inviscid Flow:

Euler Equation/Some Basic, Plane Potential Flows

(Sections 6.5-6.7)

6.4 Inviscid Flow

6.4.1 Euler's Equation of Motion

- For an inviscid flow in which the shearing stresses are all zero, and the normal stresses are replaced by -*p*, thus the equation of motion becomes

$$\rho g_x - \frac{\partial p}{\partial x} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \upsilon\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

$$\rho g_y - \frac{\partial p}{\partial y} = \rho\left(\frac{\partial \upsilon}{\partial t} + u\frac{\partial \upsilon}{\partial x} + \upsilon\frac{\partial \upsilon}{\partial y} + w\frac{\partial \upsilon}{\partial z}\right)$$

$$\rho g_z - \frac{\partial p}{\partial z} = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + \upsilon\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

or

$$\rho\vec{g} - \nabla p = \rho\left[\frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\nabla)\vec{V}\right]$$

- The main difficulty in solving the equation is the nonlinear terms which appear in the convective acceleration.

6.4.2 The Bernoulli Equation

- For steady flow

$$\rho\vec{g}-\nabla p=\rho\left(\vec{V}\cdot\nabla\right)\vec{V}$$

$$\vec{g}=-g\nabla z \quad \text{(up being positive)}$$

$$\left(\vec{V}\cdot\nabla\right)\vec{V}=\frac{1}{2}\nabla\left(\vec{V}\cdot\vec{V}\right)-\vec{V}\times\left(\nabla\times\vec{V}\right)$$

thus the equation can be written as,

$$-\rho g\nabla z-\nabla p=\frac{\rho}{2}\nabla\left(\vec{V}\cdot\vec{V}\right)-\rho\vec{V}\times\left(\nabla\times\vec{V}\right)$$

or

$$\frac{\nabla p}{\rho}+\frac{1}{2}\nabla\left(V^2\right)+g\nabla z=\vec{V}\times\left(\nabla\times\vec{V}\right)$$

- Take the dot product of each term with a differential length *ds* along a streamline

$$\frac{\nabla p}{\rho}\cdot d\vec{s}+\frac{1}{2}\nabla\left(V^2\right)\cdot d\vec{s}+g\nabla z\cdot d\vec{s}=\left[\vec{V}\times\left(\nabla\times\vec{V}\right)\right]\cdot d\vec{s}$$

- Since $d\vec{s}$ and $\vec{V}$ are parallel, therefore

$$\left[\vec{V}\times\left(\nabla\times\vec{V}\right)\right]\cdot d\vec{s}=0$$

- Since

$$d\vec{s}=dx\,\vec{i}+dy\,\vec{j}+dz\,\vec{k}$$

$$\nabla p\cdot d\vec{s}=\frac{\partial p}{\partial x}dx+\frac{\partial p}{\partial y}dy+\frac{\partial p}{\partial z}dz=dp$$

Thus the equation becomes

$$\frac{dp}{\rho}+\frac{1}{2}d\left(V^2\right)+g\,dz=0$$

where the change in $p,\vec{V},$ and z is along the streamline

- Equation after integration become

$$\int \frac{dp}{\rho} + \frac{V^2}{2} + gz = \text{constant}$$

which indicates that the sum of the three terms on the left side of the equation must remain a constant along a given streamline.

For inviscid, incompressible flow, the equation become,

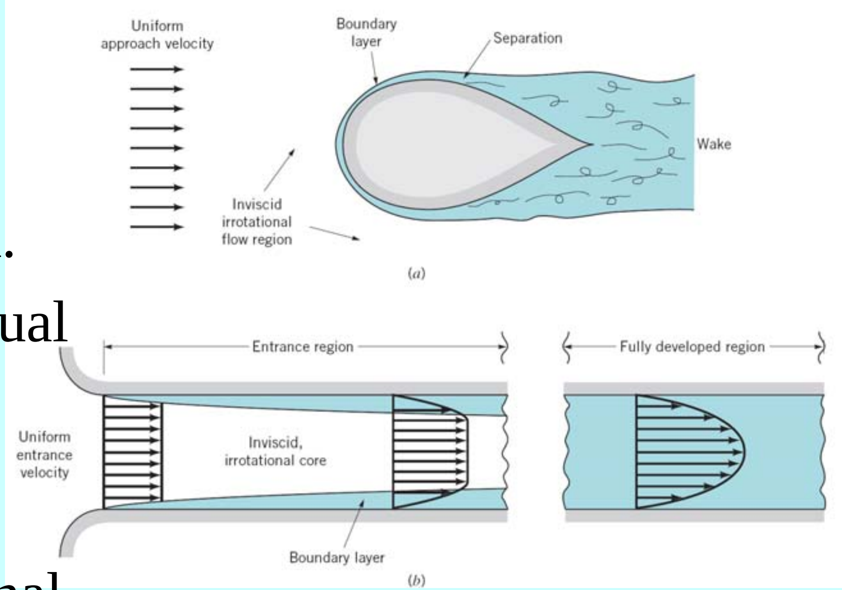
$$\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{const}$$

or

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2$$

※ For (1) inviscid flow
(2) steady flow
(3) incompressible flow
(4) flow along a streamline

6.4.3 Irrotational Flow



- If the flow is **irrotational**, the analysis of inviscid flow problem is further simplified.
- The **rotation** $\vec{\omega}$ of the fluid element is equal to $\frac{1}{2}\nabla\times\vec{V}$, and for irrotational flow field,

$$\nabla\times\vec{V}=0$$

Since $\nabla\times\vec{V}=\vec{\zeta}$, therefore for an irrotational flow field, the vorticity is zero.

- The condition of irrotationality imposes specific relationships among these velocity gradients.

For example,

$$\omega_z=\frac{1}{2}\left(\frac{\partial \upsilon}{\partial x}-\frac{\partial u}{\partial y}\right)=0$$

$$\frac{\partial \upsilon}{\partial x}=\frac{\partial u}{\partial y} \quad , \quad \frac{\partial w}{\partial y}=\frac{\partial \upsilon}{\partial z} \quad , \quad \frac{\partial u}{\partial z}=\frac{\partial w}{\partial x}$$

- A general flow field would not satisfy these three equations.

Can irrotational flow hold in a viscous fluid?

According to the 2-D ***vorticity transport equation*** (cf. Problem 6.81)

$$\frac{D\zeta_z}{Dt} = \nu\nabla^2\zeta_z$$

- Vorticity of an fluid element grows along with its motion as long as ν is positive. So, an initially irrotatioal flow will eventually turn into rotational flow in a viscous fluid.
- On the other hand, an initially irrotatioal flow remains irrotational in an inviscid fluid, if without external excitement.

6.4.4 The Bernoulli Equation for Irrotational Flow

- In Section 6.4.2, we have obtained along a streamline that,

$$\left[\vec{V}\times\left(\nabla\times\vec{V}\right)\right]\cdot ds=0$$

In an irrotational flow, $\nabla\times\vec{V}=0$, so the equation is zero regardless of the direction of $d\vec{s}$.

- Consequently, for *irrotational flow* the Bernoulli equation is valid *throughout the flow field*. Therefore, *between any flow points* in the flow field,

$$\int\frac{dp}{\rho}+\frac{V^2}{2}+gz=\text{constant}$$

or

$$\frac{p_1}{\gamma}+\frac{V_1^2}{2g}+z_1=\frac{p_2}{\gamma}+\frac{V_2^2}{2g}+z_2$$

※ For (1) Inviscid flow (2) Stead flow (3) Incompressible flow (4) Irrotational flow

6.4.5 The Velocity Potential

- For irrotational flow since

$$\nabla \times \vec{V} = 0 \quad thus \quad \vec{V} = \nabla \phi$$

$$u = \frac{\partial \phi}{\partial x} \quad , \quad \upsilon = \frac{\partial \phi}{\partial y} \quad , \quad w = \frac{\partial \phi}{\partial z}$$

  so that for an irrotational flow the velocity is expressible as the gradient of a scalar function ϕ .

- The ***velocity potential*** is a consequence of the irrotationality of the flow field (*only valid for inviscid flow*), whereas the ***stream function*** is a consequence of conservation of mass (*valid for inviscid or viscous flow*).

- **Velocity potential** can be defined for a general three-dimensional flow, whereas the **stream function** is restricted to two-dimensional flows.

- Thus for irrotational flow

$\nabla \times \vec{V} = 0 \quad \vec{V} = \nabla \phi$, further with $\nabla \cdot \vec{V} = 0$ for incomp. flow,

$\Rightarrow \nabla^2 \phi = 0$

In Cartesian coordinates, $\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} + \dfrac{\partial^2 \phi}{\partial z^2} = 0$

- Thus, inviscid, incompressible, irrotational flow fields are governed by **Laplace's equation**.
- Cylindrical coordinate

$$\nabla(\ ) = \frac{\partial(\ )}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial(\ )}{\partial \theta}\vec{e}_\theta + \frac{\partial(\ )}{\partial z}\vec{e}_z$$

$$\nabla \phi = \frac{\partial \phi}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial \phi}{\partial \theta}\vec{e}_\theta + \frac{\partial \phi}{\partial z}\vec{e}_z$$

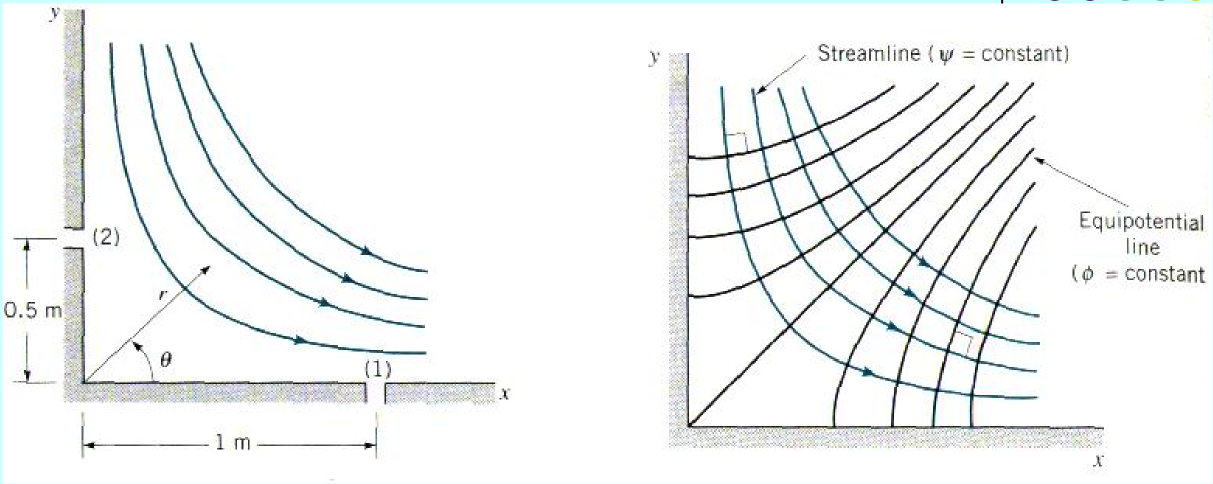
where $\phi = \phi(r, \theta, z)$

Since $\vec{V} = \upsilon_r \vec{e}_r + \upsilon_\theta \vec{e}_\theta + \upsilon_z \vec{e}_z$

Thus for an irrotational flow with $\vec{V} = \nabla \phi$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

Example 6.4



$$\psi = 2r^2 \sin 2\theta$$

$$\upsilon_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = 4r\cos 2\theta = \frac{\partial \phi}{\partial r} \quad \phi = 2r^2 \cos 2\theta + f_1(\theta)$$

$$\upsilon_\theta = -\frac{\partial \psi}{\partial r} = -4r\sin 2\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} \quad \phi = 2r^2 \cos 2\theta + f_2(r)$$

$$Thus \quad \phi = 2r^2 \cos 2\theta + C$$

The specific value of C is not important, therefore

$$\phi = 2r^2 \cos 2\theta$$

$$V^2 = (4r\cos 2\theta)^2 + (-4r\sin 2\theta)^2 = 16r^2$$

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} = \frac{p_2}{\gamma} + \frac{V_2^2}{2g}$$

6.5 Some basic, plane potential flows

- Since the Laplace equation is a linear differential equation, various solutions can be added to obtain other solutions. i.e.

$$\phi = \phi_1 + \phi_2$$

- The practical implication is that if we have basic solutions, we can combine them to obtain more complicated and interesting solutions.
- In this section several basic velocity potentials, which describe some relatively simple flows, will be determined.

- For simplicity, only two-dimensional flows will be considered.

velocity potential : $u = \dfrac{\partial \phi}{\partial x}, \quad v = \dfrac{\partial \phi}{\partial y}$ or $v_r = \dfrac{\partial \phi}{\partial r}, \quad v_\theta = \dfrac{1}{r}\dfrac{\partial \phi}{\partial \theta}$

stream function : $u = \dfrac{\partial \psi}{\partial y}, \quad v = -\dfrac{\partial \psi}{\partial x}$ or $v_r = \dfrac{1}{r}\dfrac{\partial \psi}{\partial \theta}, \quad v_\theta = -\dfrac{\partial \psi}{\partial r}$

- Defining the velocities in terms of the stream function, conservation of mass is identically satisfied.

  Now impose the condition of irrotationality,

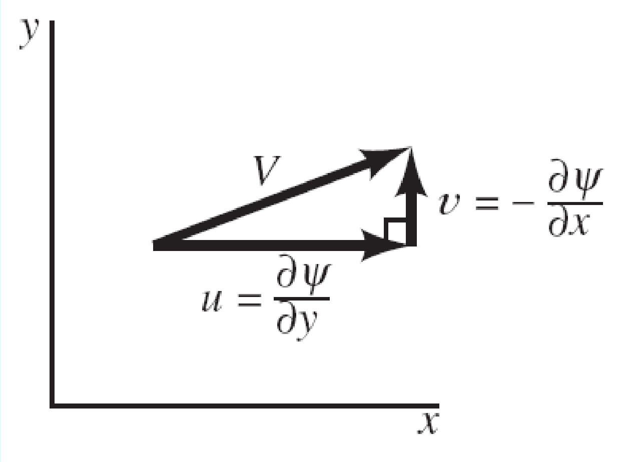
$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

Thus

$$\frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right) = \frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right) \quad \text{or} \quad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

- Thus for a two-dimensional irrotational flow, the velocity potential and the stream function both satisfy Laplace equation.
- It is apparent from these results that the velocity potential and the stream function are somehow related.

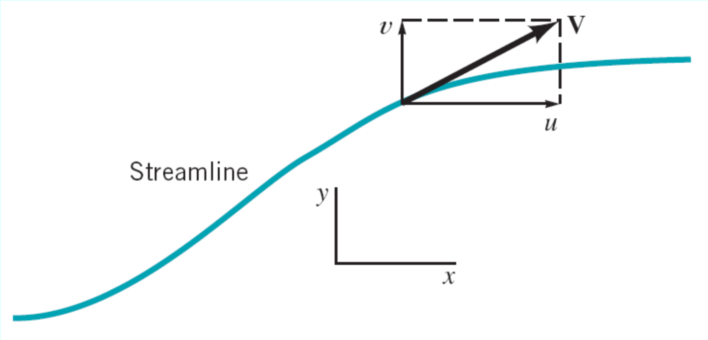
Along a line of constant ψ, $d\psi = 0$



$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = -v dx + u dy$$

$$\therefore \quad u dy = v dx, \quad \frac{dy}{dx} = \frac{v}{u}$$

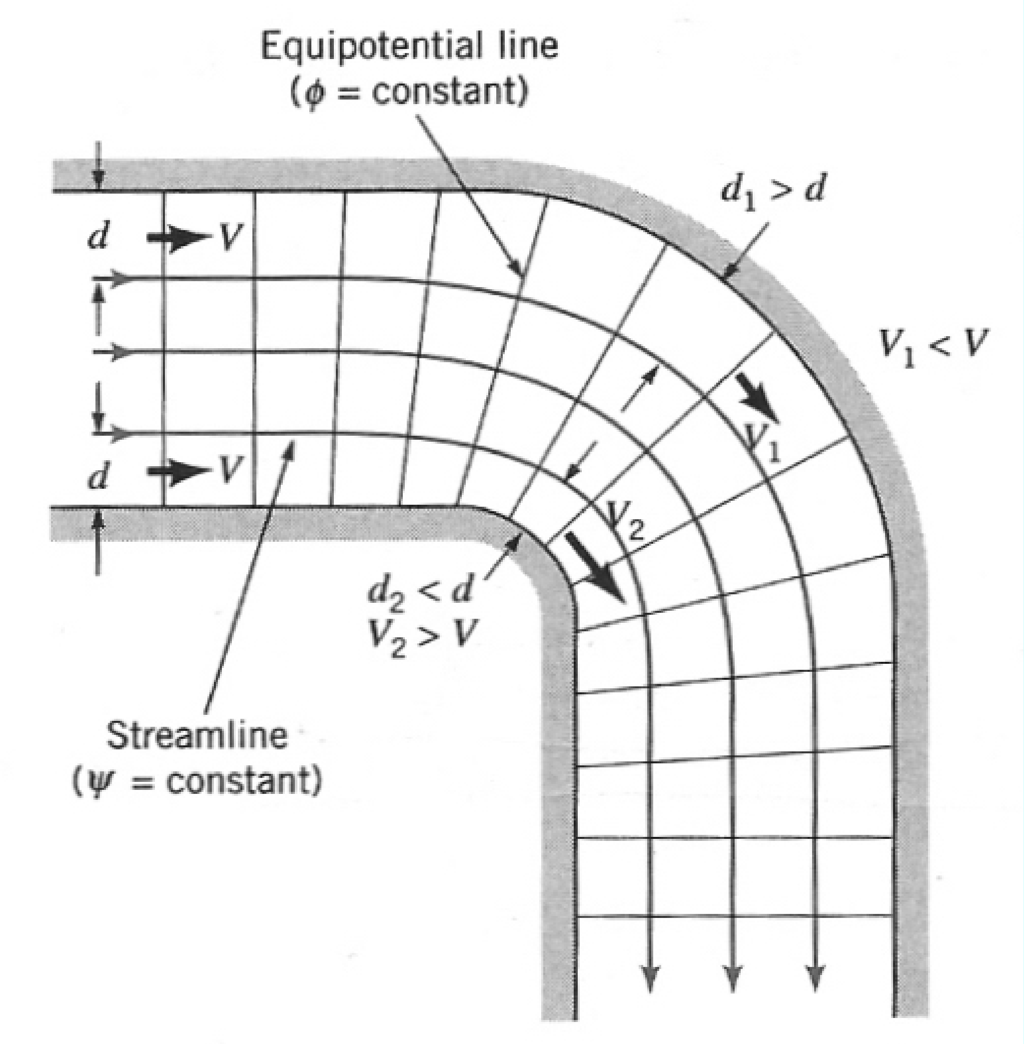
Along a line of constant ϕ, $d\phi = 0$



$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = u dx + v dy = 0$$

$$\therefore \quad u dx = -v dy, \quad \frac{dy}{dx} = -\frac{u}{v}$$

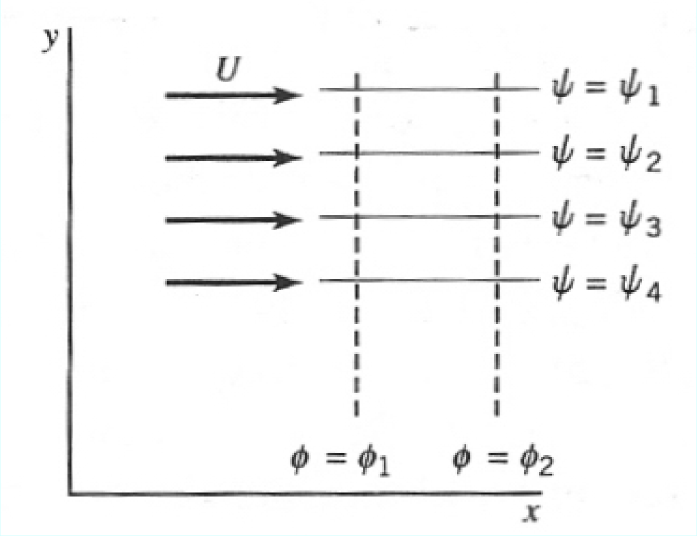
- Therefore, the equations indicate that lines of constant ϕ (**equipotential lines**) are orthogonal to lines of constant ψ (**stream line**) at all points where they intersect.



Q: Why $V_2 > V_1$?
How about p_1 and p_2?

6.5.1 Uniform Flow

- The simplest plane flow is one for which the streamlines are all straight and parallel, and the magnitude of the velocity is constant – uniform flow.



$$u = U \quad v = 0$$

$$\frac{\partial \phi}{\partial x} = U, \quad \frac{\partial \phi}{\partial y} = 0$$

$$\phi = Ux + C$$

- Thus, for a uniform flow in the positive x direction,

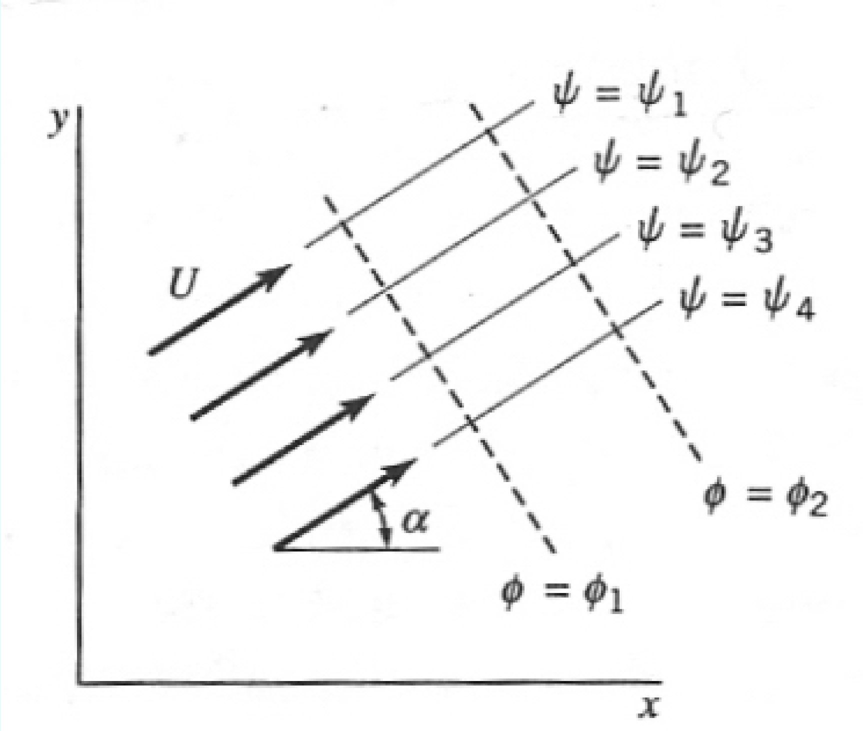
$$\phi = Ux$$

- The corresponding stream function can be obtained in a similar manner,

$$\frac{\partial \psi}{\partial y} = U, \quad \frac{\partial \psi}{\partial x} = 0 \quad \rightarrow \quad \psi = Uy$$

- The velocity potential and stream function for a uniform flow at an angle α with the x axis,

$$\phi = U(x\cos\alpha + y\sin\alpha)$$
$$\psi = U(y\cos\alpha - x\sin\alpha)$$



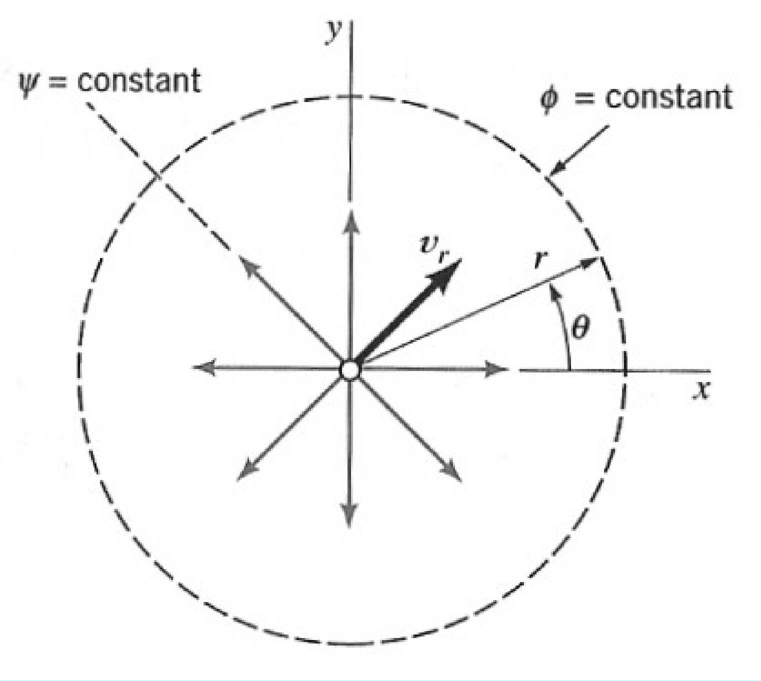
6.5.2 Source and Sink- purely radial flow

- Consider a fluid flowing radially outward from a line through the origin perpendicular to the *x*-*y* plane.

  Let *m* be the volume rate of flow emanating from the line (per unit length).

Conservation of mass

$$2\pi r\left(v_r\right)=m \quad \text{or} \quad v_r=\frac{m}{2\pi r}$$



- Also, since the flow is purely radial, velocity potential becomes, $v_\theta = 0$

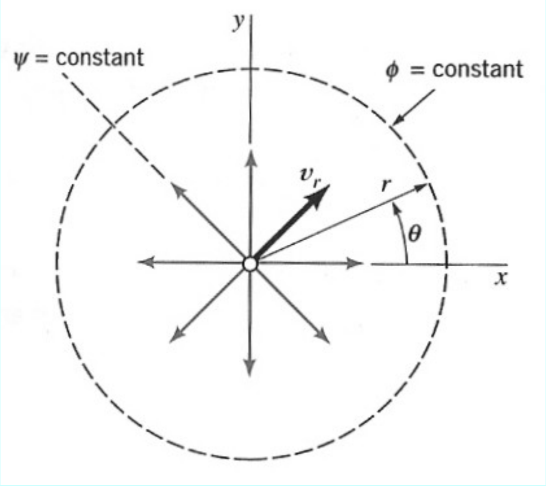
$$\frac{\partial\phi}{\partial r}=\frac{m}{2\pi r} \quad , \quad \frac{1}{r}\frac{\partial\phi}{\partial\theta}=0$$

$$\phi=\frac{m}{2\pi}\ln r$$

Source and Sink flows

- If m is positive, the flow is radially outward, and the flow is considered to be a *source* flow.
- If m is negative, the flow is toward the origin, and the flow is considered to be a *sink* flow.
- The flow rate, m, is the strength of the source or sink.
- The stream function for the source:

$$\frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{m}{2\pi r}, \quad \frac{\partial \psi}{\partial r} = 0 \quad \rightarrow \quad \psi = \frac{m}{2\pi}\theta$$



Note: At r=0, the velocity becomes infinite, which is of course physically impossible and is a singular point.

6.5.3 Vortex-streamlines are concentric circles (v_r=0)

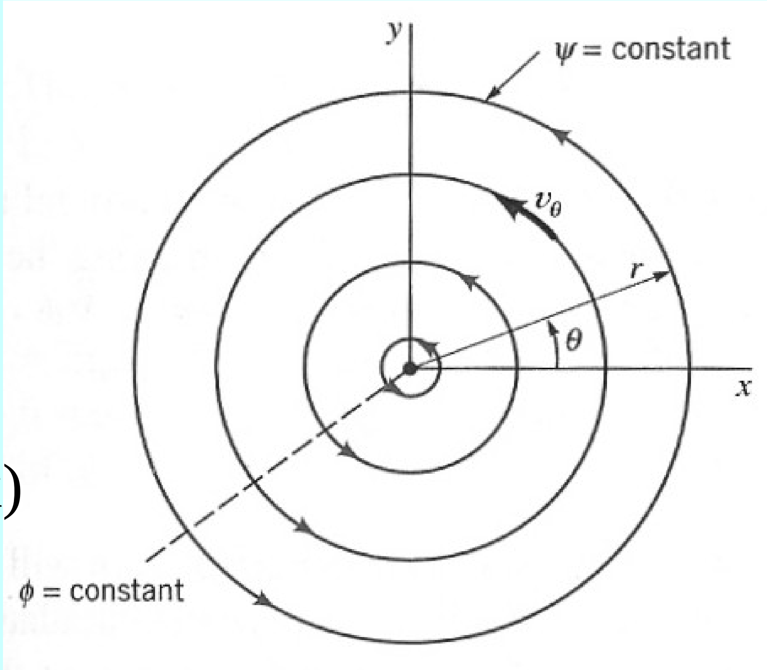
- Consider a flow field in which the streamlines are concentric circles. i.e. we interchange the velocity potential and stream function for the source.

Thus, let

$$\phi = K\theta \quad \text{and} \quad \psi = -K \ln r$$

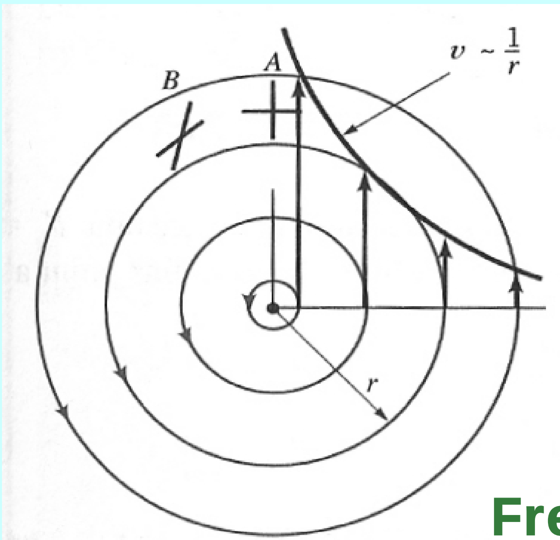
where K is a constant.

$$v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r} = \frac{K}{r} \quad \text{(free vortex)}$$

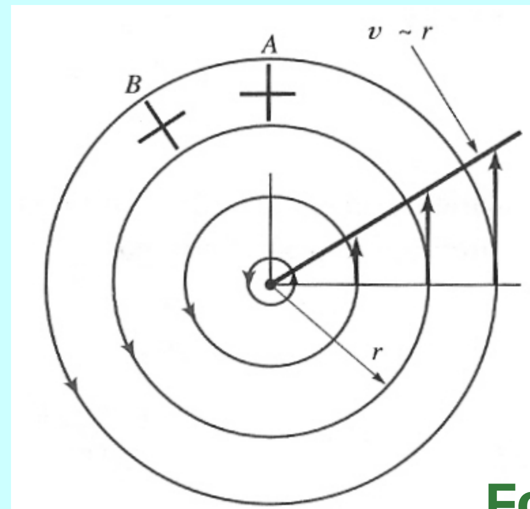


Free and Forced vortex

- Rotation refers to the orientation of a fluid element and not the path followed by the element.



Free vortex



Forced vortex

- If the fluid were rotating as a rigid body, such that $v_\theta = Kr$, this type of vortex motion is rotational and can not be described by a velocity potential.

- Free vortex: bathtub flow. V6.4 Vortex in a beaker
- Forced vortex: liquid contained in a tank rotating about its axis.

Combined vortex

- Combined vortex: a forced vortex as a central core and a free vortex outside the core.

$$v_\theta = \omega r \quad r \le r_0$$

$$v_\theta = \frac{K}{r} \quad r > r_0$$

where K and r are constant and r_0 corresponds to the radius of central core.

Circulation

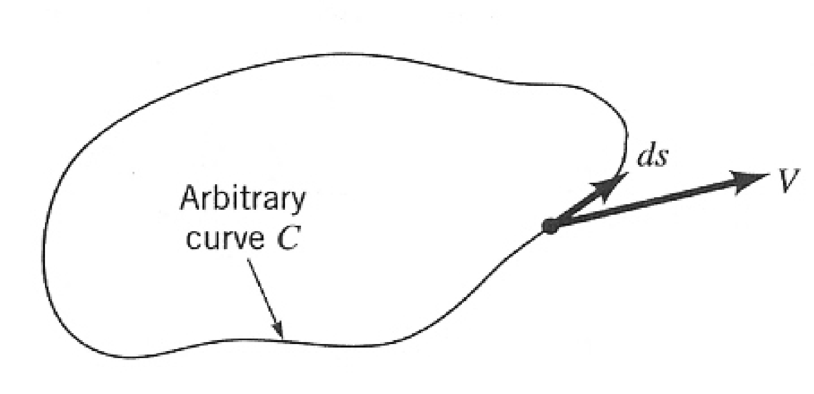
- A mathematical concept commonly associated with vortex motion is that of **circulation**.

$$\Gamma = \oint_C \vec{V} \cdot d\vec{s} \qquad (6.89)$$

The integral is taken around curve, C, in the counterclockwise direction.

Note: **Green's theorem** in the plane dictates

$$\iint_R (\nabla \times \vec{V}) \cdot \mathbf{k}\, dxdy = \oint_C \vec{V} \cdot d\vec{s}$$



- For an irrotational flow

$$\vec{V} = \nabla\phi \rightarrow \vec{V} \cdot d\vec{s} = \nabla\phi \cdot d\vec{s} = d\phi$$

therefore,

$$\Gamma = \oint_C d\phi = 0$$

For an irrotational flow, the circulation will generally be zero. However, if there are singularities enclosed within the curve, the circulation may not be zero.

Circulation for free vortex

- For example, the free vortex with $v_\theta = \dfrac{K}{r}$

$$\Gamma = \int_0^{2\pi} \frac{K}{r}\left(r d\theta\right) = 2\pi K \qquad K = \frac{\Gamma}{2\pi}$$

Note: However Γ along any path which does not include the singular point at the origin will be zero.

- The velocity potential and stream function for the free vortex are commonly expressed in terms of circulation as,

$$\phi = \frac{\Gamma}{2\pi}\theta \tag{6.90}$$

$$\psi = -\frac{\Gamma}{2\pi}\ln r \tag{6.91}$$

Example 6.6

Determine an expression relating the surface shape to the strength of the vortex as specified by circulation Γ.

$$\phi = \frac{\Gamma}{2\pi}\theta$$

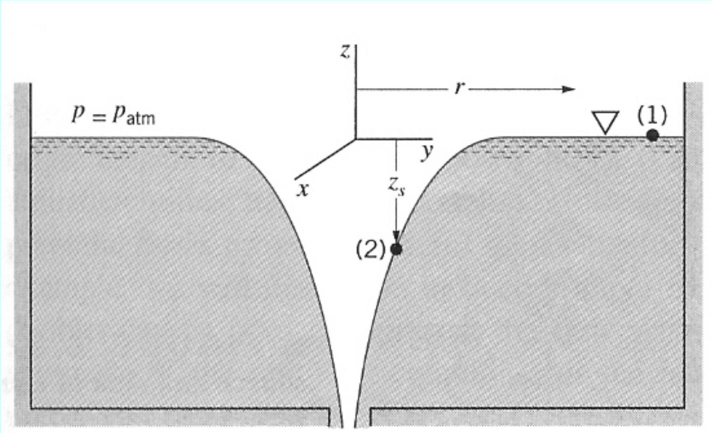
- For irrotational flow, the Bernoulli equation

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2 \qquad p_1 = p_2 = 0$$

$$\frac{V_1^2}{2g} = z_s + \frac{V_2^2}{2g} \qquad z_1 = 0$$

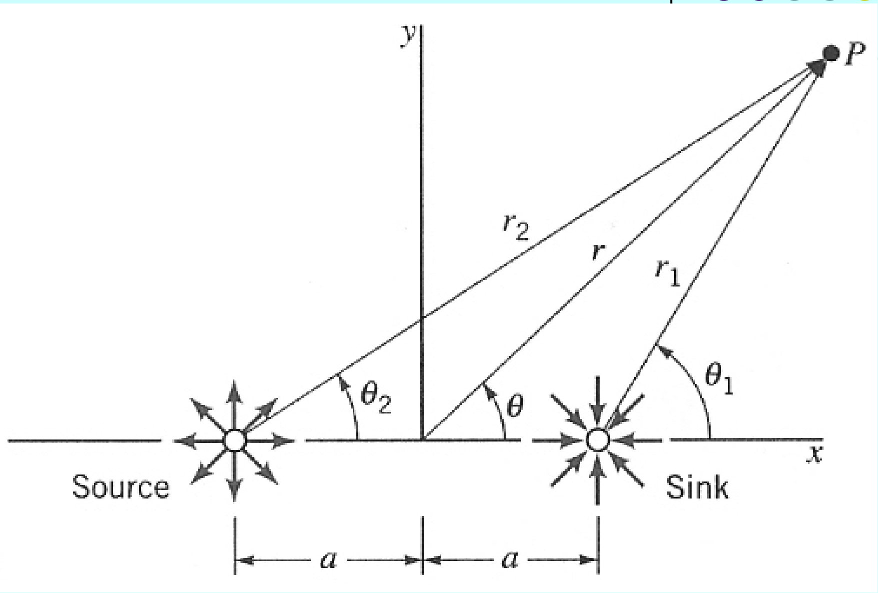
$$v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{\Gamma}{2\pi r} \qquad V_1 \approx 0$$

$$z_s = -\frac{\Gamma^2}{8\pi^2 r^2 g}$$



6.5.4 Doublet

- Consider potential flow that is formed by combining a source and a sink in a special way.
- Consider a source-sink pair



$$\psi = -\frac{m}{2\pi}(\theta_1 - \theta_2) \quad \rightarrow \quad \tan\left(-\frac{2\pi\psi}{m}\right) = \tan(\theta_1 - \theta_2) = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2}$$

Since $\quad \tan\theta_1 = \dfrac{r\sin\theta}{r\cos\theta - a} \quad$ and $\quad \tan\theta_2 = \dfrac{r\sin\theta}{r\cos\theta + a}$

Thus $\quad \tan\left(-\dfrac{2\pi\psi}{m}\right) = \dfrac{2ar\sin\theta}{r^2 - a^2} \quad \rightarrow \quad \psi = -\dfrac{m}{2\pi}\tan^{-1}\left(\dfrac{2ar\sin\theta}{r^2 - a^2}\right)$

Doublet

For small values of a

$$\psi = -\frac{m}{2\pi}\frac{2ar\sin\theta}{r^2-a^2} = -\frac{mar\sin\theta}{\pi(r^2-a^2)} \tag{6.94}$$

- **Doublet** is formed by letting the source and sink approach one another ($a \to 0$) while increasing the strength m ($m \to \infty$) so that the product ma/π remains constant.

  As $a \to 0, \quad r/(r^2-a^2) \to 1/r$

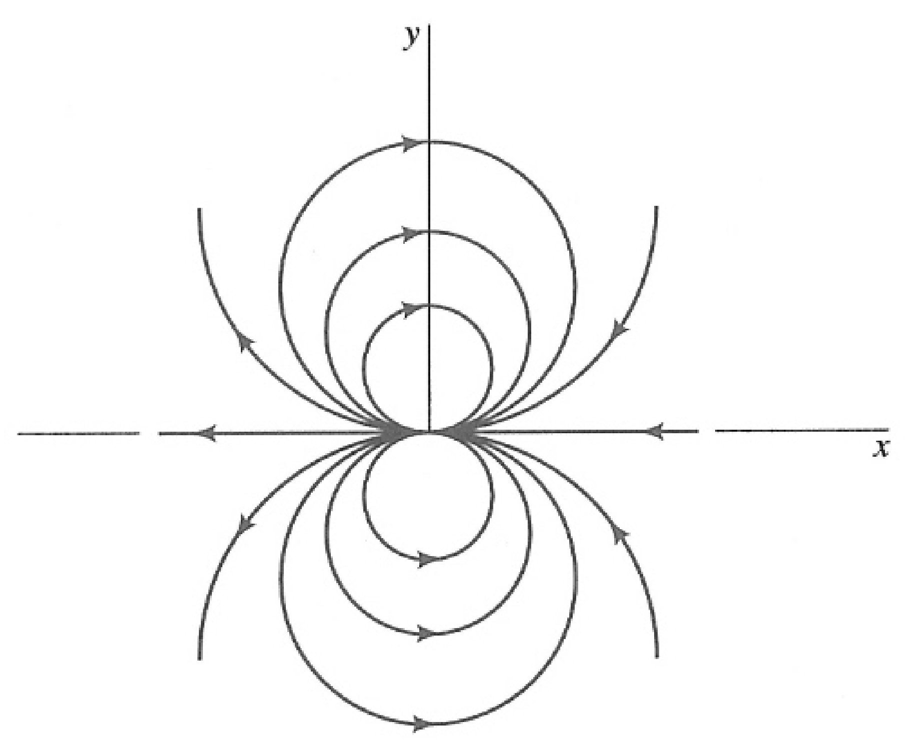
  Eq. 6.94 reduces to:
  $$\psi = -\frac{K\sin\theta}{r} \tag{6.95}$$

  where $K = ma/\pi$ is called the *strength* of the doublet.

- The corresponding velocity potential is

$$\phi = \frac{K\cos\theta}{r} \tag{6.96}$$

Doublet-streamlines



$$V_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{K\cos\theta}{r^2}$$

$$V_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r} = -\frac{K\sin\theta}{r^2}$$

Streamlines for a doublet

- Summary of basic, plane potential flows

TABLE 6.1

Summary of Basic, Plane Potential Flows

Description of Flow Field	Velocity Potential	Stream Function	Velocity Components[a]
Uniform flow at angle α with the x axis (see **Fig. 6.16*b***)	$\phi = U(x\cos\alpha + y\sin\alpha)$	$\psi = U(y\cos\alpha - x\sin\alpha)$	$u = U\cos\alpha$ $v = U\sin\alpha$
Source or sink (see **Fig. 6.17**) $m > 0$ source $m < 0$ sink	$\phi = \frac{m}{2\pi}\ln r$	$\psi = \frac{m}{2\pi}\theta$	$v_r = \frac{m}{2\pi r}$ $v_\theta = 0$
Free vortex (see **Fig. 6.18**) $\Gamma > 0$ counterclockwise motion $\Gamma < 0$ clockwise motion	$\phi = \frac{\Gamma}{2\pi}\theta$	$\psi = -\frac{\Gamma}{2\pi}\ln r$	$v_r = 0$ $v_\theta = \frac{\Gamma}{2\pi r}$
Doublet (see **Fig. 6.23**)	$\phi = \frac{K\cos\theta}{r}$	$\psi = -\frac{K\sin\theta}{r}$	$v_r = -\frac{K\cos\theta}{r^2}$ $v_\theta = -\frac{K\sin\theta}{r^2}$

[a]Velocity components are related to the velocity potential and stream function through the relationships:

$$u = \frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} \qquad v = \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x} \qquad v_r = \frac{\partial\phi}{\partial r} = \frac{1}{r}\frac{\partial\psi}{\partial\theta} \qquad v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{\partial\psi}{\partial r}.$$

6.6 Superposition of Basic, Plane Potential Flows

Method of superposition

- Any streamline in an inviscid flow field can be considered as a solid boundary, since the conditions along a solid boundary and a streamline are the same-no flow through the boundary or the streamline.
- Therefore, some basic velocity potential or stream function can be combined to yield a streamline that corresponds to a particular body shape of interest.
- This method is called the **method of superposition**.

6.6.1 Source in a Uniform Stream- Half-Body

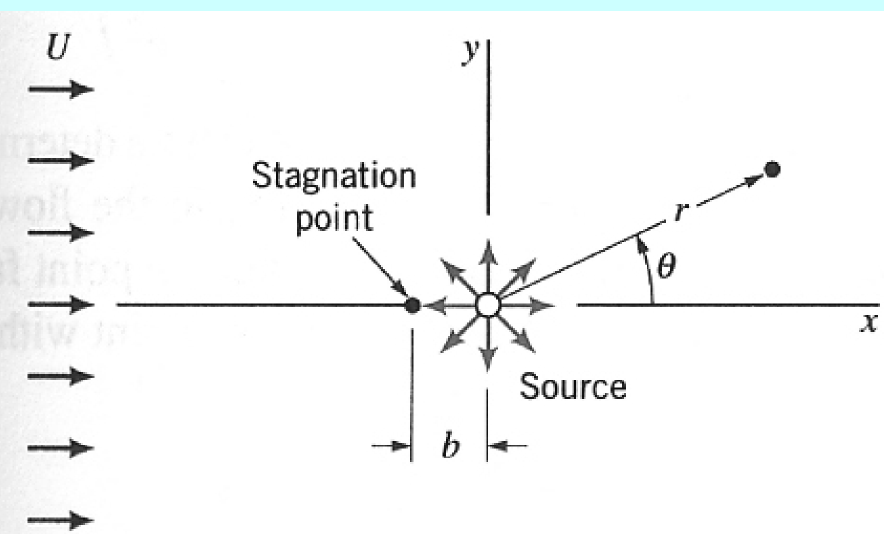
- Consider a superposition of a source and a uniform flow.
- The resulting stream function is

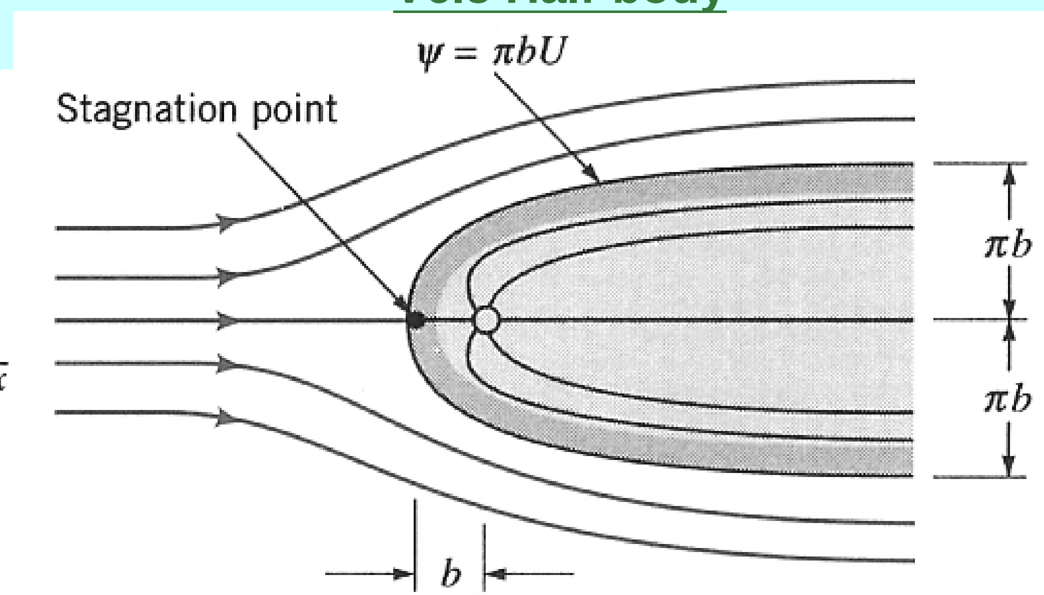
$$\psi = \psi_{\text{uniform flow}} + \psi_{\text{source}}$$
$$= Ur\sin\theta + \frac{m}{2\pi}\theta \qquad (6.97)$$

and the corresponding velocity potential is

$$\phi = Ur\cos\theta + \frac{m}{2\pi}\ln r$$

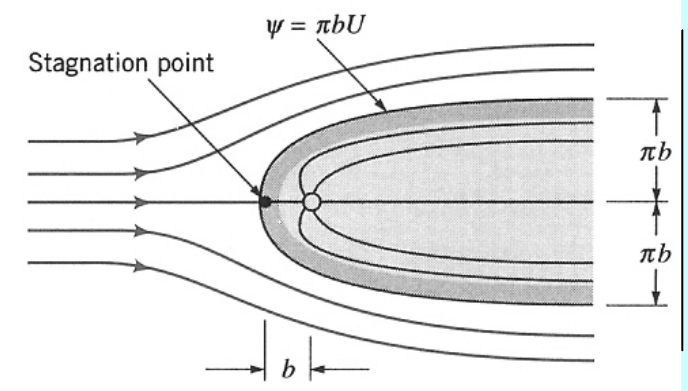
V6.5 Half-body





- For the source alone

$$v_r = \frac{m}{2\pi r}$$



Let the stagnation point occur at $x = -b$, where $U = \dfrac{m}{2\pi b}$

so $$b = \frac{m}{2\pi U}$$

The value of the stream function at the stagnation point can be obtained by evaluating x at $r=b$ and $\theta=\pi$, which yields from Eq. 6.97

$$\psi_{\text{stagnation}} = \frac{m}{2} = \pi b U$$

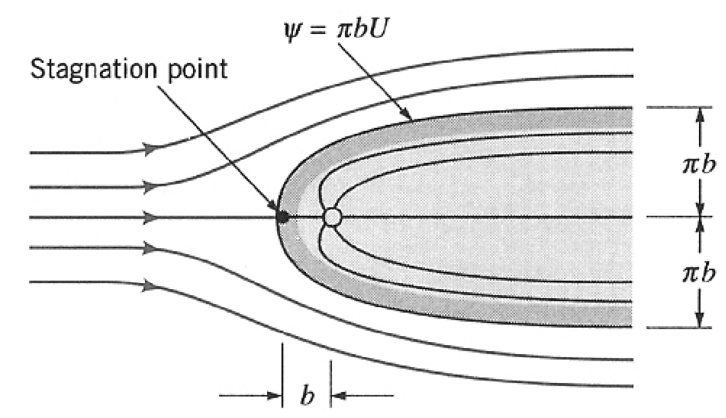
Thus the equation of the streamline passing through the stagnation point is,

$$\pi b U = U r \sin\theta + b U \theta \qquad \text{or} \qquad r = \frac{b(\pi - \theta)}{\sin\theta} \qquad (6.100)$$

- The width of the half-body asymptotically approaches $2\pi b$. This follows from Eq. 6.100, which can be written as

$$y = b(\pi - \theta)$$

so that as $\theta \to 0$ or $\theta \to 2\pi$, the half-width approaches $\pm b\pi$.



- With the stream function (or velocity potential) known, the velocity components at any point can be obtained.

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = U\cos\theta + \frac{m}{2\pi r}$$

$$v_\theta = -\frac{\partial \psi}{\partial r} = -U\sin\theta$$

- Thus the square of the magnitude of the velocity V at any point is,

$$V^2 = v_r^{\,2} + v_\theta^{\,2} = U^2 + \frac{Um\cos\theta}{\pi r} + \left(\frac{m}{2\pi r}\right)^2$$

$$\text{since } b = \frac{m}{2\pi U}$$

$$V^2 = U^2\left(1 + 2\frac{b}{r}\cos\theta + \frac{b^2}{r^2}\right) \qquad (6.101)$$

- With the velocity known, the pressure distribution can be determined from the Bernoulli equation,

$$p_0 + \frac{1}{2}\rho U^2 = p + \frac{1}{2}\rho V^2 \qquad (6.102)$$

Note: the velocity tangent to the surface of the body is not zero; that is, the fluid slips by the boundary.

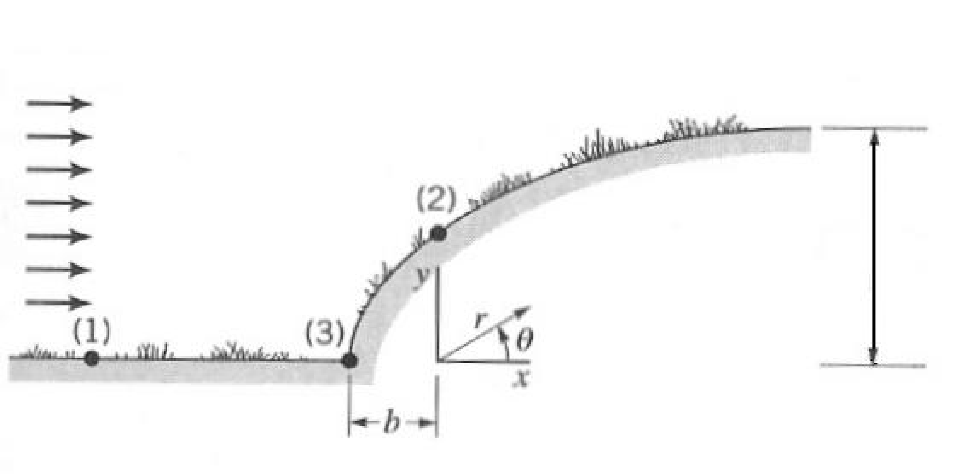
Example 6.7

$$V^2 = U^2\left(1 + 2\frac{b}{r}\cos\theta + \frac{b^2}{r^2}\right)$$

on the surface $\theta = \pi/2$ $\quad r = \dfrac{b(\pi - \theta)}{\sin\theta} = \dfrac{\pi b}{2}$

Thus $\quad V^2 = U^2\left(1 + \dfrac{4}{\pi^2}\right)$

$$y_2 = \frac{\pi b}{2}$$



$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + y_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + y_2$$

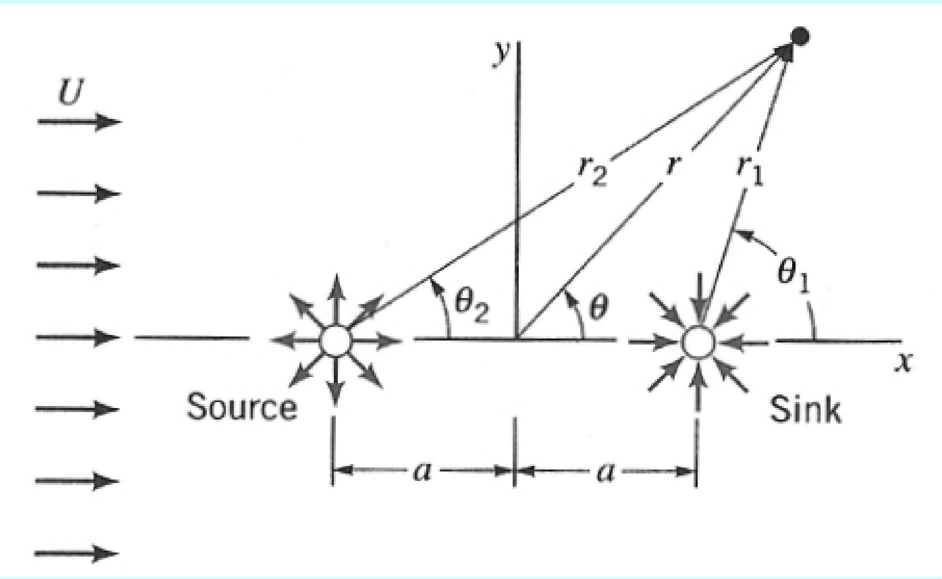
$$p_1 - p_2 = \frac{\rho}{2}\left(V_2^2 - V_1^2\right) + \gamma\left(y_2 - y_1\right)$$

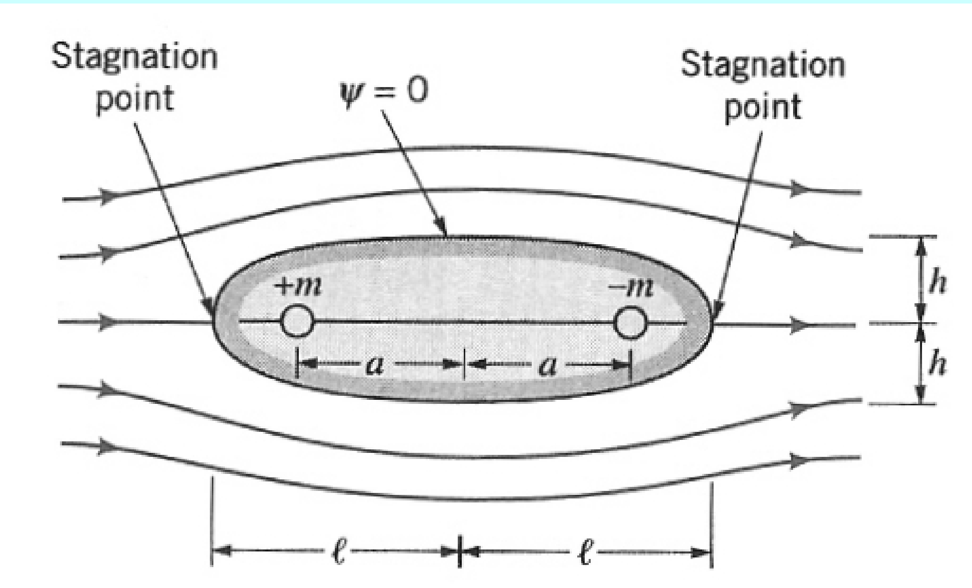
6.6.2 Rankine Ovals

- Consider a source and a sink of equal strength combined with a uniform flow to form the flow around a closed body.

  The stream function and velocity potential for this combination are,

$$\psi = Ur\sin\theta - \frac{m}{2\pi}(\theta_1 - \theta_2) \qquad \phi = Ur\cos\theta - \frac{m}{2\pi}(\ln r_1 - \ln r_2)$$

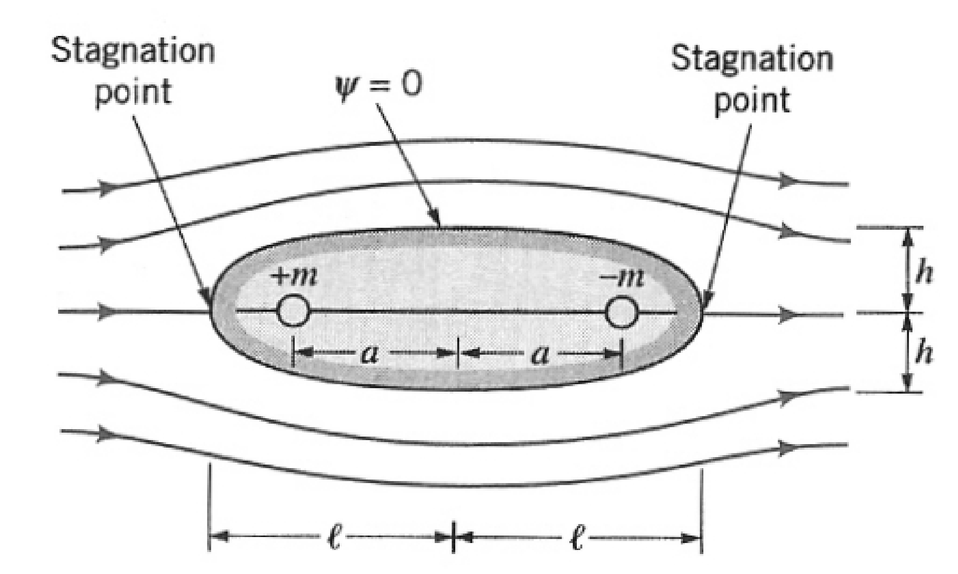




As in Section 6.5.4

$$\psi = Ur\sin\theta - \frac{m}{2\pi}\tan^{-1}\left(\frac{2ar\sin\theta}{r^2 - a^2}\right)$$

or $$\psi = Uy - \frac{m}{2\pi}\tan^{-1}\left(\frac{2ay}{x^2 + y^2 - a^2}\right)$$



The stream line ψ=0 forms a closed body.

Since the body is closed, all of the flow emanating from the source flows into the sink.

- These bodies have an oval shape and are termed **Rankine ovals**.
- The stagnation points correspond to the points where the uniform velocity, the source velocity, and the sink velocity all combine to give a zero velocity.
- The location of the stagnation points depend on the value of a, m , and U.

- The body half length:

$$l=\left(\frac{ma}{\pi U}+a^2\right)^{1/2} \quad \text{or} \quad \frac{l}{a}=\left(\frac{m}{\pi Ua}+1\right)^{1/2}$$

source: $$v_r=\frac{m}{2\pi r}$$

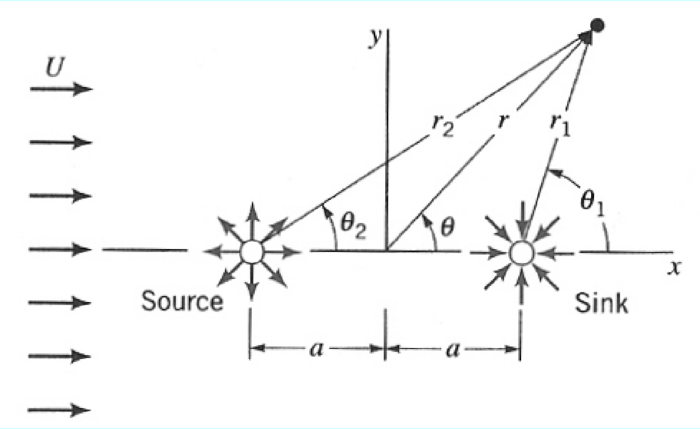
Therefore

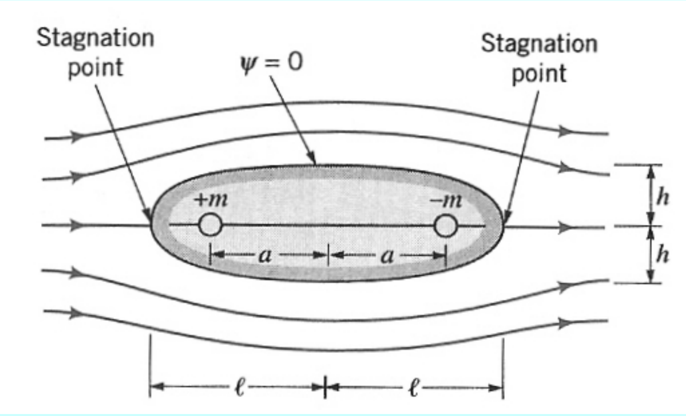
$$U-\frac{m}{2\pi(r-a)}+\frac{m}{2\pi(r+a)}=0$$

$$U-\frac{m}{2\pi}\frac{2a}{r^2-a^2}=0$$

$$1-\frac{m}{\pi U}\frac{1}{r^2-a^2}=0 \quad \text{or} \quad r^2-a^2=\frac{m}{\pi U}$$

$$l=r=\left(\frac{m}{\pi U}+a^2\right)^{1/2}$$





- The body half width, h, can be obtained by determining the value of y where the y axis intersects the $\psi=0$ streamline. Thus, from Eq. 6.105 with $\psi=0$, $x=0$, and $y=h$, It follows that

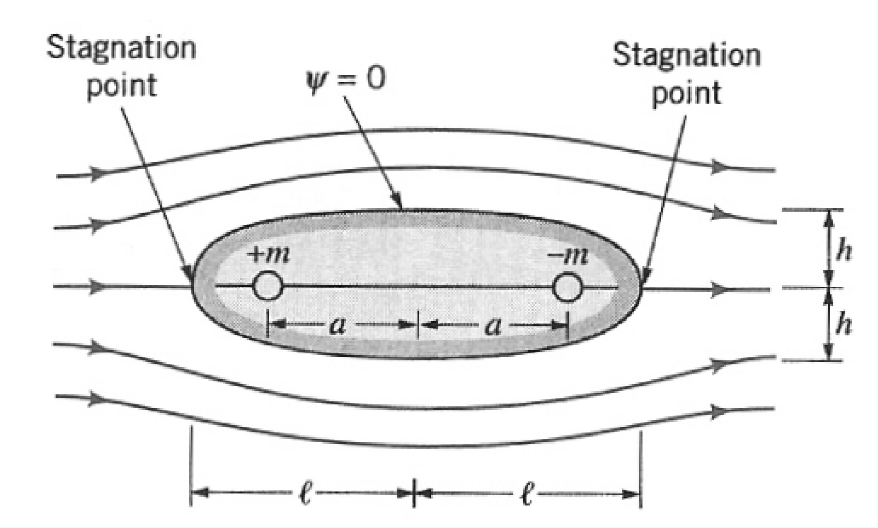
$$\psi = Uy - \frac{m}{2\pi}\tan^{-1}\left(\frac{2ay}{x^2+y^2-a^2}\right) \quad \rightarrow \quad 0 = Uh - \frac{m}{2\pi}\tan^{-1}\left(\frac{2ah}{h^2-a^2}\right)$$

$$\tan^{-1}\left(\frac{2ah}{h^2-a^2}\right) = \frac{Uh2\pi}{m}$$

$$\frac{2ah}{h^2-a^2} = \tan\frac{2\pi Uh}{m}$$

$$h = \frac{h^2-a^2}{2a}\tan\frac{2\pi Uh}{m}$$

$$\frac{h}{a} = \frac{1}{2}\left[\left(\frac{h}{a}\right)^2 - 1\right]\tan\frac{2\pi Uh}{m} = \frac{1}{2}\left[\left(\frac{h}{a}\right)^2 - 1\right]\tan\left[2\left(\frac{\pi Ua}{m}\right)\frac{h}{a}\right]$$



- Both *l/a* and *h/a* are functions of the dimensionless parameter *Ua/m*.
- As *l/h* becomes large, flow around a long slender body is described, whereas for small value of parameter, flow around a more blunt shape is obtained.
- Downstream from the point of maximum body width the surface pressure increase with distance along the surface. In actual *viscous* flow, an adverse pressure gradient will lead to *separation* of the flow from the surface and result in a large low pressure wake on the downstream side of the body.
- However, separation is not predicted by potential theory.
- Rankine ovals will give a reasonable approximation of the velocity outside the thin, viscous boundary layer and the pressure distribution on the front part of the body.

V6.6 Circular cylinder
V6.8 Circular cylinder with separation
V6.9 Potential and viscous flow

6.6.3 Flow around a circular cylinder

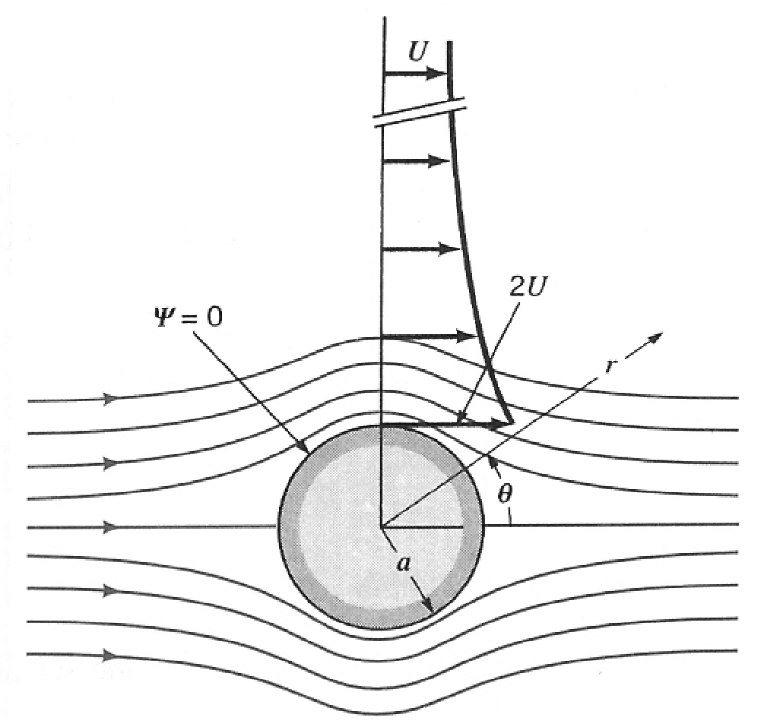
- When the distance between the source-sink pair approaches zero, the shape of the Rankine oval becomes more blunt and approach a circular shape.
- A combination of doublet and uniform flow will represent flow around a circular cylinder.

$$\text{stream function:}\quad \psi = Ur\sin\theta - \frac{K\sin\theta}{r} = \left(U - \frac{K}{r^2}\right) r\sin\theta$$

$$\text{velocity potential:}\quad \phi = Ur\cos\theta + \frac{K\cos\theta}{r}$$

to determine K with $\psi=0$ for $r=a$,

$$U - \frac{K}{a^2} = 0 \quad \rightarrow \quad K = Ua^2$$



- Thus the stream function and velocity potential for flow around a circular cylinder are

$$\psi = Ur\left(1-\frac{a^2}{r^2}\right)\sin\theta$$

$$\phi = Ur\left(1+\frac{a^2}{r^2}\right)\cos\theta$$

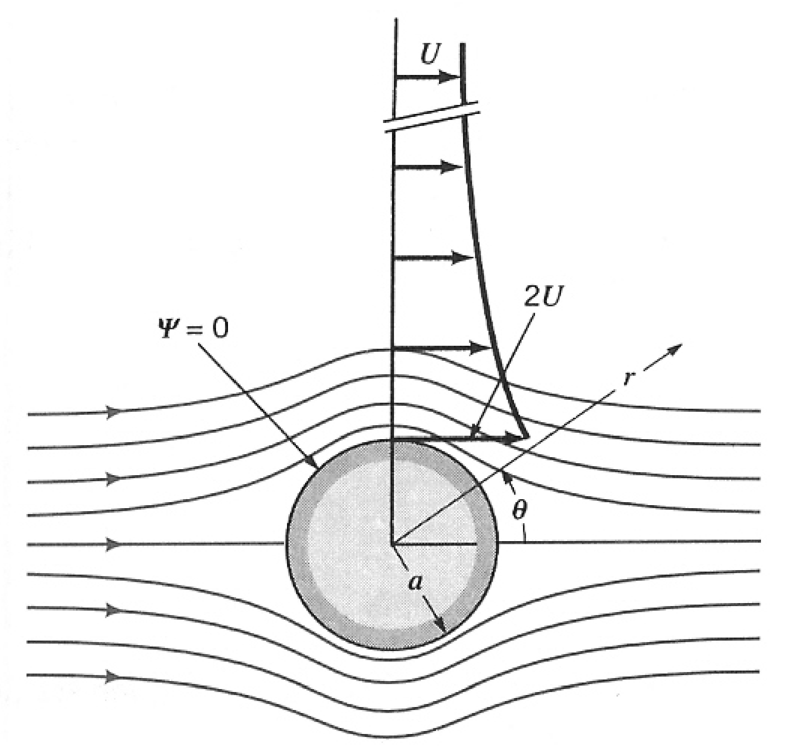
- The velocity components are

$$v_r = \frac{\partial\phi}{\partial r} = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = U\left(1-\frac{a^2}{r^2}\right)\sin\theta$$

$$v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{\partial\psi}{\partial r} = -U\left(1+\frac{a^2}{r^2}\right)\sin\theta$$

- On the cylinder surface (*r*=*a*):

$$v_r = 0 \quad \text{and} \quad v_\theta = -2U\sin\theta$$



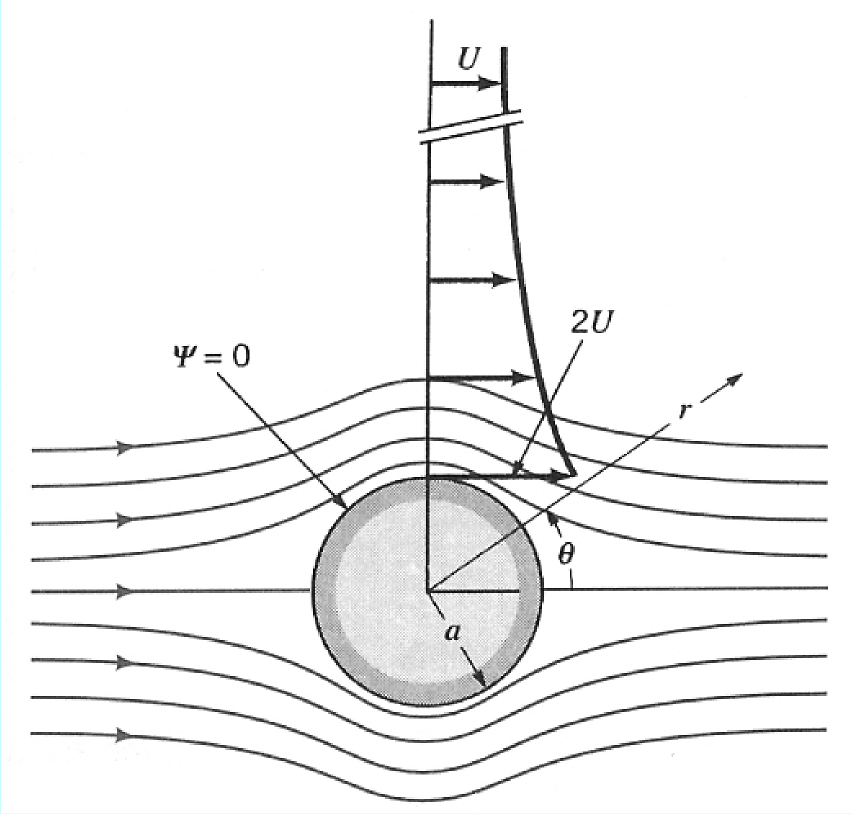
Potential flow around a circular cylinder

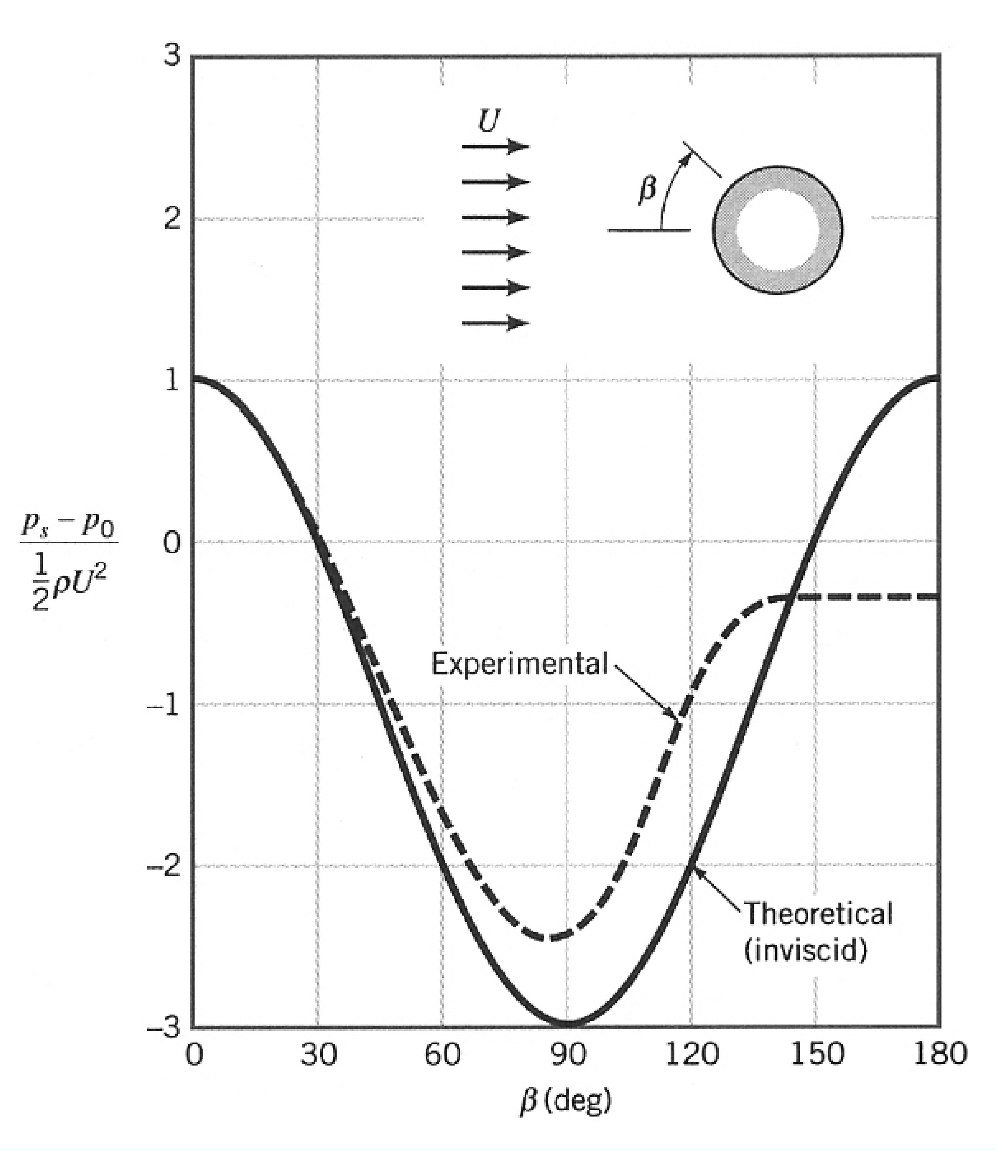
- Therefore the maximum velocity occurs at the top and bottom of the cylinder $\theta = \pm\pi/2$ and has a magnitude of twice the upstream velocity U.
- The pressure distribution on the cylinder surface is obtained from the Bernoulli equation,

$$p_0 + \frac{1}{2}\rho U^2 = p_s + \frac{1}{2}\rho {v_{\theta s}}^2$$

$$p_s = p_0 + \frac{1}{2}\rho U^2\left(1 - 4\sin^2\theta\right)$$

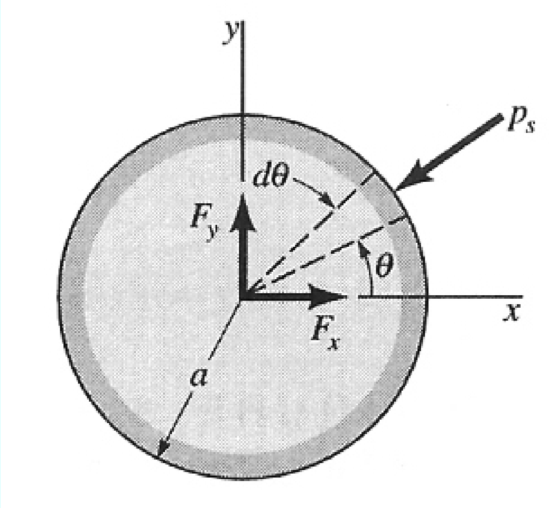
where p_0 and U are pressure and velocity for point far from the cylinder.





- The figure reveals that only on the upstream part of the cylinder is there approximate agreement between the potential flow and the experimental results.

- The resulting force (per unit length) developed on the cylinder can be determined by integrating the pressure over the surface.



$$F_x = -\int_0^{2\pi} p_s \cos\theta a d\theta = 0$$

$$F_y = -\int_0^{2\pi} p_s \sin\theta a d\theta = 0$$

- *Both the drag and lift as predicted by potential theory for a fixed cylinder in a uniform stream are zero*. since the pressure distribution is *symmetrical* around the cylinder.
- In reality, there is a significant drag developed on a cylinder when it is placed in a moving fluid. (**d'Alembert paradox**)

Ex 6.8 Potential flow--cylinder

- By adding a free vortex to the stream function or velocity potential for the flow around a cylinder, then

$$\psi = Ur\left(1-\frac{a^2}{r^2}\right)\sin\theta - \frac{\Gamma}{2\pi}\ln r \tag{6.119}$$

$$\phi = Ur\left(1+\frac{a^2}{r^2}\right)\cos\theta + \frac{\Gamma}{2\pi}\theta \tag{6.120}$$

where Γ is the circulation

- Tangential velocity on the surface (r=a):

$$v_{\theta s} = -\left.\frac{\partial\psi}{\partial r}\right|_{r=a} = -2U\sin\theta + \frac{\Gamma}{2\pi a} \tag{6.121}$$

- This type of flow could be approximately created by placing a rotating cylinder in a uniform stream. Because the presence of viscosity in any real fluid, the fluid in contact with the rotating cylinder would rotate with the same velocity as the cylinder, and the resulting flow field would resemble that developed by the combination of a uniform flow past a cylinder and a free vortex.

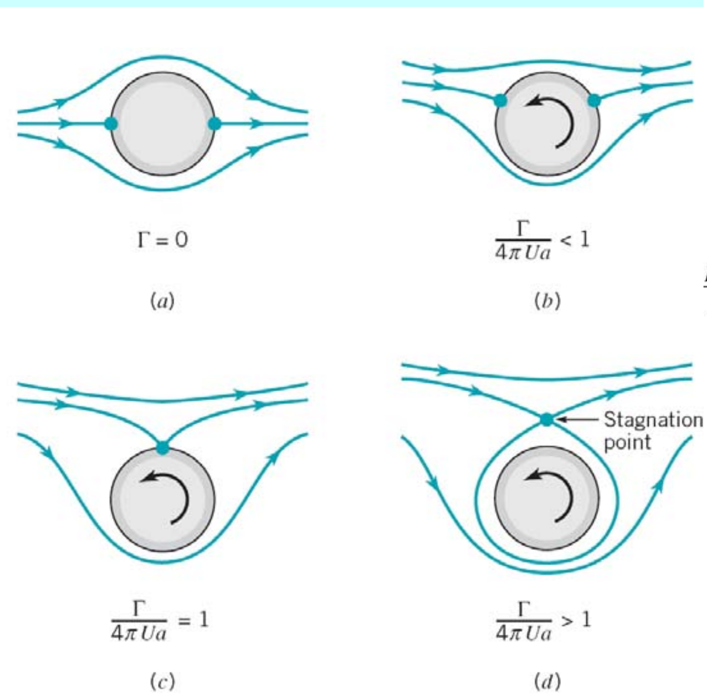
- Location of the stagnation point

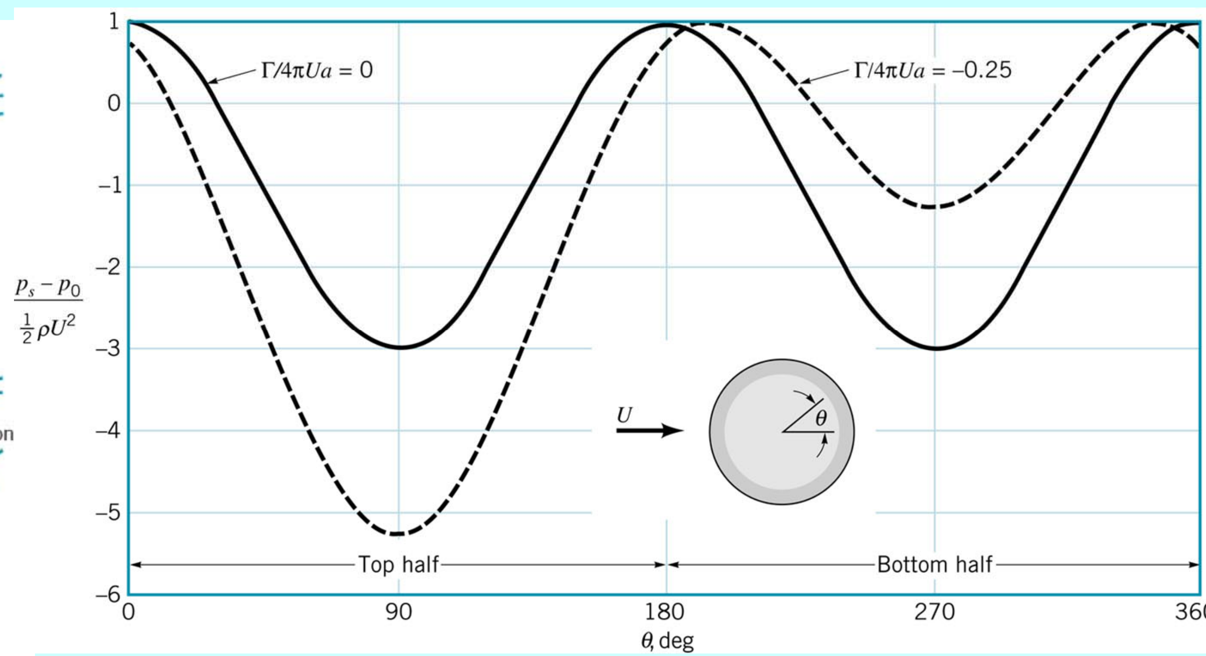
$$v_{\theta s} = 0 = -2U\sin\theta + \frac{\Gamma}{2\pi a} \quad \rightarrow \quad \sin\theta\Big|_{\text{stag}} = \frac{\Gamma}{4\pi U a}$$

$$\text{if} \quad \Gamma = 0 \ \rightarrow \ \theta\Big|_{\text{stag}} = 0 \ \text{ or } \ \pi$$

if $-1 \le \Gamma / 4\pi U a \le 1 \rightarrow \theta\Big|_{\text{stag}}$ is at some other location on the surface

if $\Gamma / 4\pi U a > 1 \rightarrow \theta\Big|_{\text{stag}}$ is located away from the cylinder





- Force per unit length developed on the cylinder

$$p_0 + \frac{1}{2}\rho U^2 = p_s + \frac{1}{2}\rho\left(-2U\sin\theta + \frac{\Gamma}{2\pi a}\right)^2$$

$$p_s = p_0 + \frac{1}{2}\rho U^2\left(1 - 4\sin^2\theta + \frac{2\Gamma\sin\theta}{\pi a U} - \frac{\Gamma^2}{4\pi^2 a^2 U^2}\right)$$

$$F_x = -\int_0^{2\pi} p_s \cos\theta a d\theta = 0$$

$$F_y = -\int_0^{2\pi} p_s \sin\theta a d\theta = -\frac{\rho U\Gamma}{\pi}\int_0^{2\pi}\sin^2\theta d\theta = -\rho U\Gamma$$

- For a cylinder with circulation, lift is developed equal to the product of the fluid density, the upstream velocity, and the circulation.

$$F_y = -\rho U\Gamma$$

$U(+)$ $\Gamma(+$, counterclockwise$)$ the F_y is downward

The development of this lift on rotating bodies is called the **Magnus effect**.

6.7 Other Aspects of Potential Flow Analysis

- Exact solutions based in potential theory will usually provide at best approximate solutions to real fluid problems.
- Potential theory will usually provide a reasonable approximation in those circumstances when we are dealing with a low viscosity fluid moving at a relatively high velocity, in regions of the flow field in which the flow is accelerating.
- Outside the boundary layer the velocity distribution and the pressure distribution are closely approximated by the potential flow solution.
- In situation when the flow is decelerating (in the rearward portion of the bluff body expanding region of a conduit), and adverse pressure gradient is reduced leading to *flow separation*, a phenomenon that are not accounted for by potential theory.

V6.10 Potential flow

PART C

Viscous Flow:

Navier-Stokes Equation

(Sections 6.8-6.10)

6.8 Viscous Flow

Equation of Motion

$$\delta \vec{F}_x = \delta m a_x \quad \delta \vec{F}_y = \delta m a_y \quad \delta \vec{F}_z = \delta m a_z$$

$$\delta m = \rho \delta x \delta y \delta z$$

Thus

$$\rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right)$$

$$\rho g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right)$$

$$\rho g_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \rho \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right)$$

6.8.1 Stress-Deformation Relationships

When a shear stress is applied on a fluid:

- Fluids continuously deform (stress τ ~ rate of strain)
- Solids deform or bend (stress τ ~ strain)

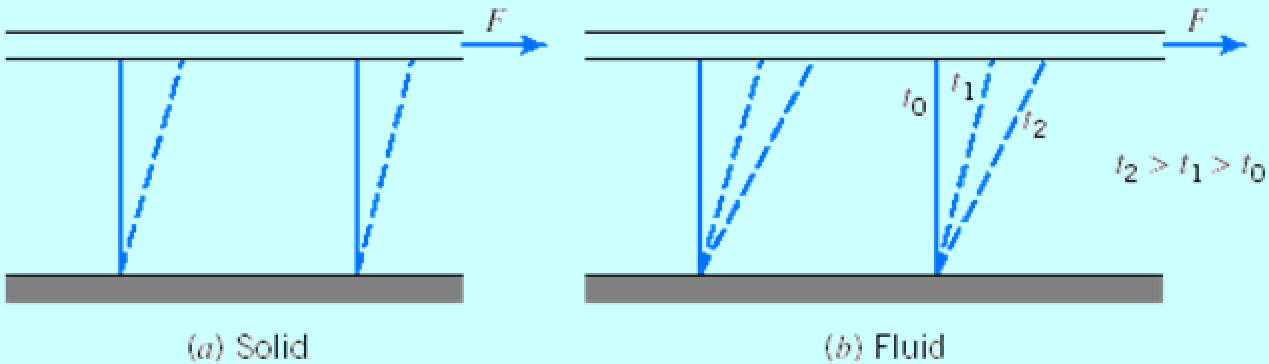


Fig. 1.1 Behavior of a solid and a fluid, under the action of a constant shear force.

strain rate ~ velocity gradient

$$\frac{d\alpha}{dt} = \frac{du}{dy}$$

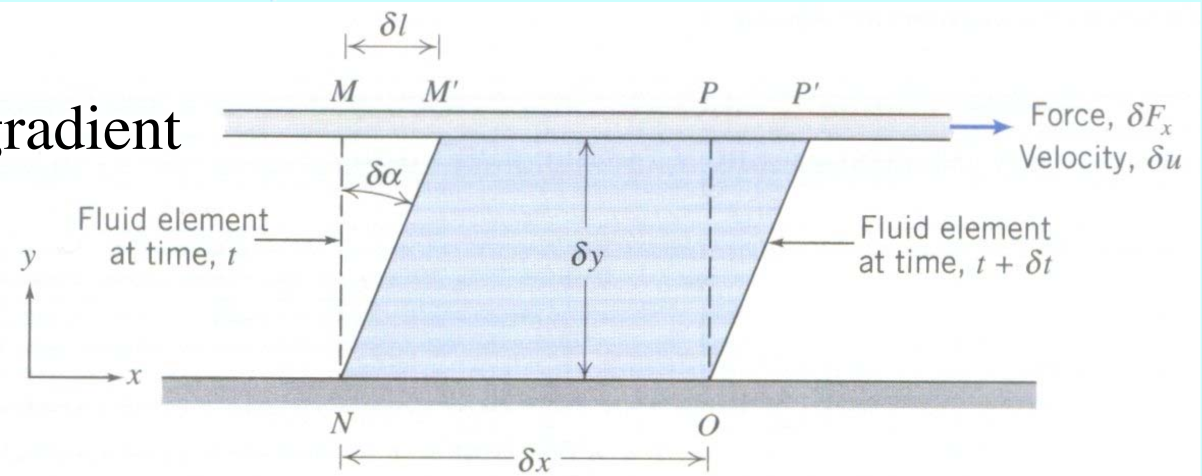


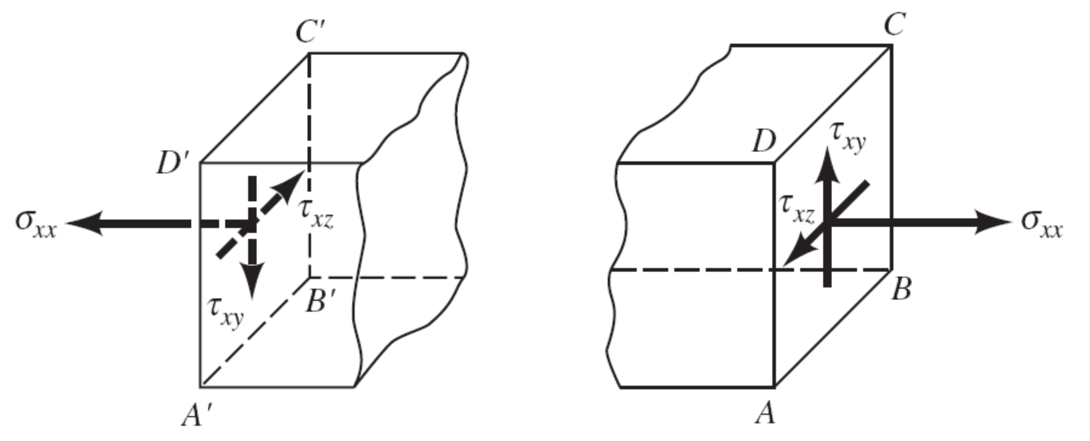
Fig. 2.7 Deformation of a fluid element.

from Fox, McDonald and Pritchard, Introduction to Fluid Mechanics.

6.8.1 Stress-Deformation Relationships

- For incompressible **Newtonian fluids** it is known that the stresses are linearly related to the rate of deformation.

V1.6 Non-Newtonian behavior



For incompressible, **Newtonian** fluids, the **viscous stresses** are:

$$\boldsymbol{\tau}_{\text{visc},ij} \equiv \begin{bmatrix} \sigma_{xx,\text{visc}} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy,\text{visc}} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz,\text{visc}} \end{bmatrix} \equiv \begin{bmatrix} 2\mu\dfrac{\partial v_x}{\partial x} & \mu\left(\dfrac{\partial v_x}{\partial y}+\dfrac{\partial v_y}{\partial x}\right) & \mu\left(\dfrac{\partial v_z}{\partial x}+\dfrac{\partial v_x}{\partial z}\right) \\ \mu\left(\dfrac{\partial v_x}{\partial y}+\dfrac{\partial v_y}{\partial x}\right) & 2\mu\dfrac{\partial v_y}{\partial y} & \mu\left(\dfrac{\partial v_y}{\partial z}+\dfrac{\partial v_z}{\partial y}\right) \\ \mu\left(\dfrac{\partial v_z}{\partial x}+\dfrac{\partial v_x}{\partial z}\right) & \mu\left(\dfrac{\partial v_y}{\partial z}+\dfrac{\partial v_z}{\partial y}\right) & 2\mu\dfrac{\partial v_z}{\partial z} \end{bmatrix}$$

6.8.1 Stress-Deformation Relationships

But in normal stresses, there is additional contribution of pressure *p*, where

$$-p = \frac{1}{3}\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right)$$

Consequently,

for normal stresses

$$\sigma_{xx} = -p + 2\mu \frac{\partial u}{\partial x}$$

$$\sigma_{yy} = -p + 2\mu \frac{\partial \upsilon}{\partial y}$$

$$\sigma_{zz} = -p + 2\mu \frac{\partial w}{\partial z}$$

for shearing stresses

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial \upsilon}{\partial x}\right)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial \upsilon}{\partial z} + \frac{\partial w}{\partial y}\right)$$

$$\tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

Can you figure out why the **normal viscous stress** $\sigma_{xx,\mathrm{visc}}$ can be expressed as $2\mu \,\partial u / \partial x$?

6.8.1 Stress-Deformation Relationships

- For viscous fluids in motion the normal stresses are not necessarily the same in different directions, thus, the need to define the pressure as the average of the three normal stresses.
- Stress-strain relationship in cylindrical coordinate

$$\sigma_{rr} = -p + 2\mu \frac{\partial \upsilon_r}{\partial r} \qquad \tau_{r\theta} = \tau_{\theta r} = \mu \left(r \frac{\partial}{\partial r}\left(\frac{\upsilon_\theta}{r}\right) + \frac{1}{r}\frac{\partial \upsilon_r}{\partial \theta} \right)$$

$$\sigma_{\theta\theta} = -p + 2\mu \left(\frac{1}{r}\frac{\partial \upsilon_\theta}{\partial \theta} + \frac{\upsilon_r}{r} \right) \qquad \tau_{\theta z} = \tau_{z\theta} = \mu \left(\frac{\partial \upsilon_\theta}{\partial z} + \frac{1}{r}\frac{\partial \upsilon_z}{\partial \theta} \right)$$

$$\sigma_{zz} = -p + 2\mu \frac{\partial \upsilon_z}{\partial z} \qquad \tau_{rz} = \tau_{zr} = \mu \left(\frac{\partial \upsilon_r}{\partial z} + \frac{\partial \upsilon_z}{\partial r} \right)$$

Note: Notation τ_{xy} x: plane perpendicular to x coordinate
y: direction

6.8.2 The Navier-Stokes Equations

- The **Navier-Stokes equations** are considered to be the governing differential equations of motion for *incompressible Newtonian fluids*

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\upsilon\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}+\rho g_x+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial \upsilon}{\partial t}+u\frac{\partial \upsilon}{\partial x}+\upsilon\frac{\partial \upsilon}{\partial y}+w\frac{\partial \upsilon}{\partial z}\right)=-\frac{\partial p}{\partial y}+\rho g_y+\mu\left(\frac{\partial^2 \upsilon}{\partial x^2}+\frac{\partial^2 \upsilon}{\partial y^2}+\frac{\partial^2 \upsilon}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+\upsilon\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\rho g_z+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

The Navier-Stokes Equations

- In terms of cylindrical coordinate

$$\rho\left(\frac{\partial \upsilon_r}{\partial t}+\upsilon_r\frac{\partial \upsilon_r}{\partial r}+\frac{\upsilon_\theta}{r}\frac{\partial \upsilon_r}{\partial \theta}-\frac{\upsilon_\theta^2}{r}+\upsilon_z\frac{\partial \upsilon_r}{\partial z}\right)$$

$$=-\frac{\partial p}{\partial r}+\rho g_r+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \upsilon_r}{\partial r}\right)-\frac{\upsilon_r}{r^2}+\frac{1}{r^2}\frac{\partial^2 \upsilon_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial \upsilon_\theta}{\partial \theta}+\frac{\partial^2 \upsilon_r}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial \upsilon_\theta}{\partial t}+\upsilon_r\frac{\partial \upsilon_\theta}{\partial r}+\frac{\upsilon_\theta}{r}\frac{\partial \upsilon_\theta}{\partial \theta}+\frac{\upsilon_r\upsilon_\theta}{r}+\upsilon_z\frac{\partial \upsilon_\theta}{\partial z}\right)$$

$$=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\rho g_\theta+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \upsilon_\theta}{\partial r}\right)-\frac{\upsilon_\theta}{r^2}+\frac{1}{r^2}\frac{\partial^2 \upsilon_\theta}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial \upsilon_r}{\partial \theta}+\frac{\partial^2 \upsilon_\theta}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial \upsilon_z}{\partial t}+\upsilon_r\frac{\partial \upsilon_z}{\partial r}+\frac{\upsilon_\theta}{r}\frac{\partial \upsilon_z}{\partial \theta}+\upsilon_z\frac{\partial \upsilon_z}{\partial z}\right)$$

$$=-\frac{\partial p}{\partial z}+\rho g_z+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \upsilon_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 \upsilon_z}{\partial \theta^2}+\frac{\partial^2 \upsilon_z}{\partial z^2}\right)$$

6.9 Some Simple Solutions for Viscous, Incompressible Fluids

- There are no general analytical schemes for solving nonlinear partial differential equations, and each problem must be considered individually.

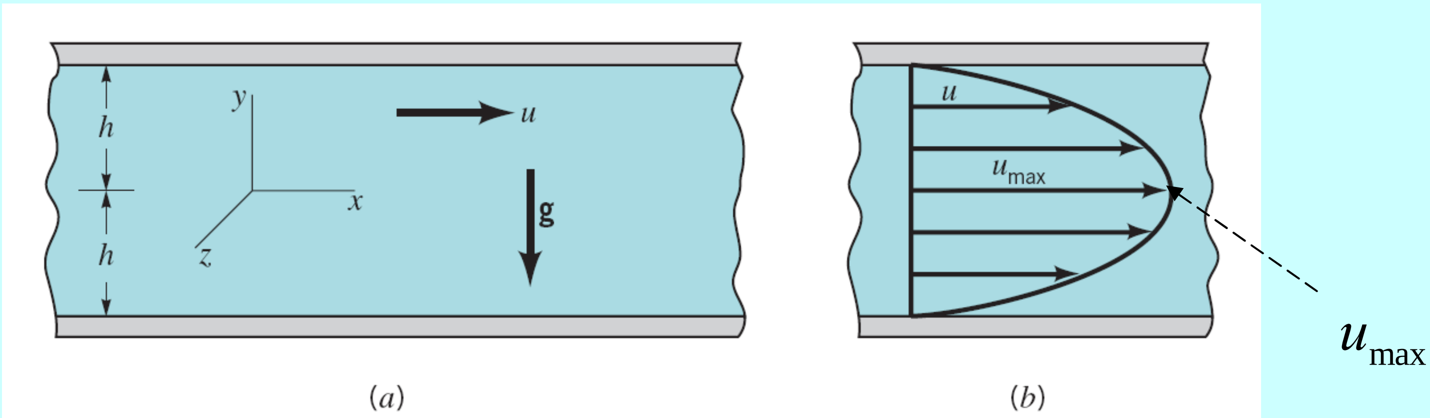
$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\upsilon\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}+\rho g_x+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial \upsilon}{\partial t}+u\frac{\partial \upsilon}{\partial x}+\upsilon\frac{\partial \upsilon}{\partial y}+w\frac{\partial \upsilon}{\partial z}\right)=-\frac{\partial p}{\partial y}+\rho g_y+\mu\left(\frac{\partial^2 \upsilon}{\partial x^2}+\frac{\partial^2 \upsilon}{\partial y^2}+\frac{\partial^2 \upsilon}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+\upsilon\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\rho g_z+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

Nonlinear terms

6.9.1 Steady Laminar Flow Between Fixed Parallel plates



$$\upsilon = 0, w = 0$$

Thus continuity indicates that

$$\frac{\partial u}{\partial x} = 0$$

for steady flow, $u = u(y)$

$$g_x = 0, g_y = -g \;\; and \;\; g_z = o$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \upsilon\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial \upsilon}{\partial t} + u\frac{\partial \upsilon}{\partial x} + \upsilon\frac{\partial \upsilon}{\partial y} + w\frac{\partial \upsilon}{\partial z}\right) = -\frac{\partial p}{\partial y} + \rho g_y + \mu\left(\frac{\partial^2 \upsilon}{\partial x^2} + \frac{\partial^2 \upsilon}{\partial y^2} + \frac{\partial^2 \upsilon}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + \upsilon\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \rho g_z + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

Steady Laminar Flow Between Fixed Parallel plates

Thus

$$0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2}$$

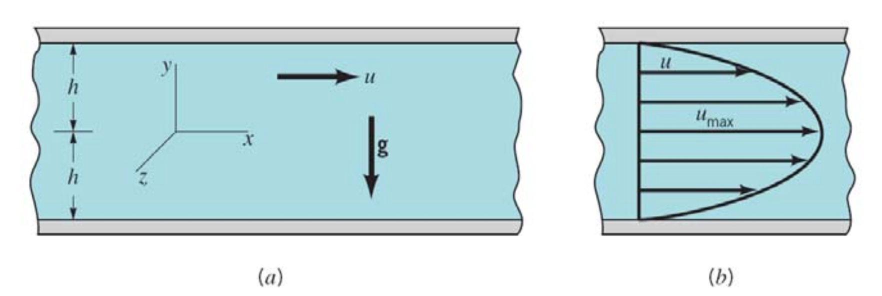
$$0 = -\frac{\partial p}{\partial y} - \rho g \;\rightarrow p = -\rho g y + f_1(x)$$

$$0 = -\frac{\partial p}{\partial z}$$

$$\frac{d^2 u}{dy^2} = \frac{1}{\mu}\frac{\partial p}{\partial x}$$

$$\frac{du}{dy} = \frac{1}{\mu}\left(\frac{\partial p}{\partial x}\right) y + C_1$$

$$u = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right) y^2 + C_1 y + C_2$$



$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \upsilon\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial \upsilon}{\partial t} + u\frac{\partial \upsilon}{\partial x} + \upsilon\frac{\partial \upsilon}{\partial y} + w\frac{\partial \upsilon}{\partial z}\right) = -\frac{\partial p}{\partial y} + \rho g_y + \mu\left(\frac{\partial^2 \upsilon}{\partial x^2} + \frac{\partial^2 \upsilon}{\partial y^2} + \frac{\partial^2 \upsilon}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + \upsilon\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \rho g_z + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

(${}^{\partial p}\!/\!_{\partial x}$ is treated as a constant since it is not a function of *y*)

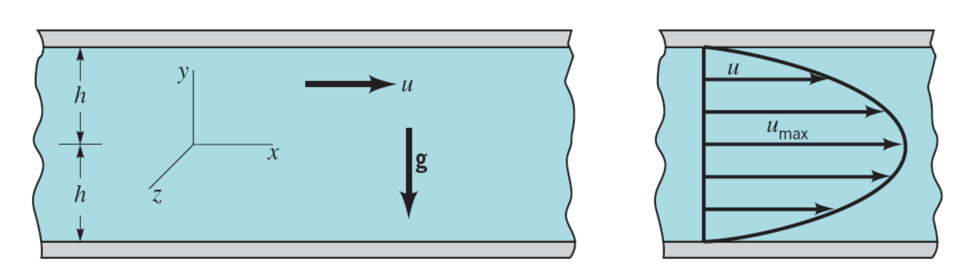
Steady Laminar Flow Between Fixed Parallel plates

the constants are determined from the boundary conditions. V6.11 No-slip boundary conditions

$$\text{BCs}: \; u = 0 \quad for \quad y = \pm h$$

$$\text{Thus} \quad C_1 = 0$$

$$C_2 = -\frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)h^2$$



(a) (b)

Thus the velocity distribution becomes,

$$u = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)\left(y^2 - h^2\right)$$

which indicates that the velocity profile between the two fixed plates is parabolic.

V6.13 Laminar flow

Steady Laminar Flow Between Fixed Parallel plates

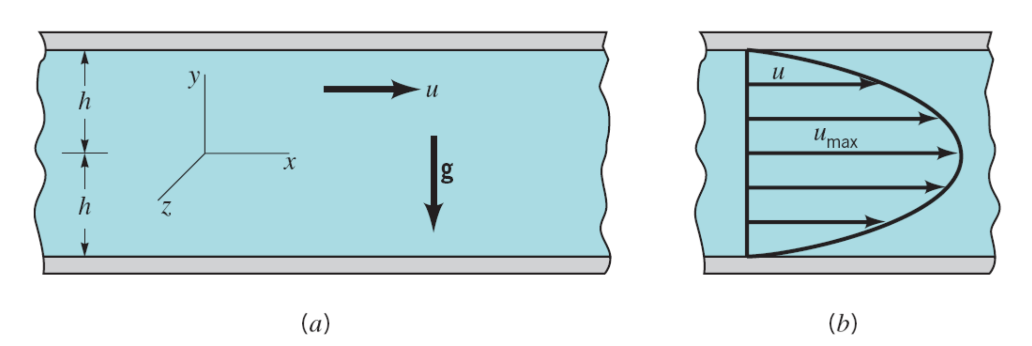
- The volume rate of flow

$$q = \int_{-h}^{h} u\, dy = \int_{-h}^{h} \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)\left(y^2 - h^2\right) dy$$

$$q = \frac{1}{2\mu}\frac{\partial p}{\partial x}\left[\left. \frac{y^3}{3} - h^2 y \right|_{-h}^{h}\right]$$

$$= \frac{1}{2\mu}\frac{\partial p}{\partial x}\left[\frac{y^3}{3} - h^3 + \frac{h^3}{3} - h^3\right]$$

$$= -\frac{2}{3}\frac{h^3}{\mu}\frac{\partial p}{\partial x}$$



The pressure gradient is negative, since the pressure decreases in the direction of the flow.

Steady Laminar Flow Between Fixed Parallel plates

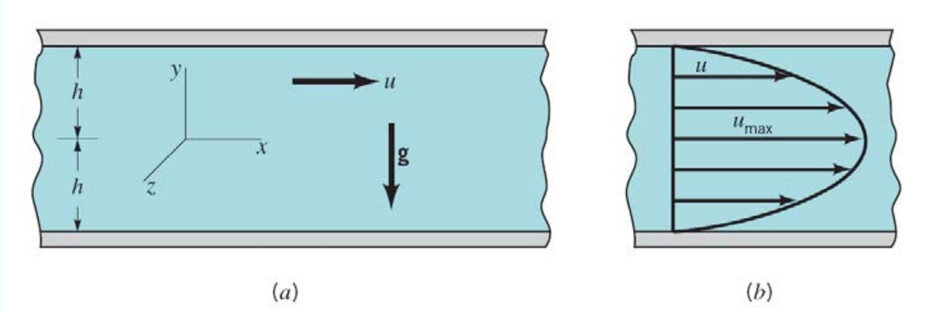
If Δp represents the pressure drop between two points a distance ℓ apart, then

$$\frac{\Delta p}{\ell} = -\frac{\partial p}{\partial x}$$

$$q = -\frac{2}{3}\frac{h^3}{\mu}\frac{\partial p}{\partial x} = \frac{2h^3 \Delta p}{3\mu\ell}, \qquad V = \frac{q}{2h} = \frac{h^2 \Delta p}{3\mu\ell}$$

- The maximum velocity u_{max} , occurs midway $y=0$ between the two plates, thus

$$u_{max} = -\frac{h^2}{2\mu}\frac{\partial p}{\partial x} \qquad \text{or} \qquad u_{max} = \frac{3}{2}V$$



Steady Laminar Flow Between Fixed Parallel plates

- The pressure field

$$p = -\rho g y + f(x)$$

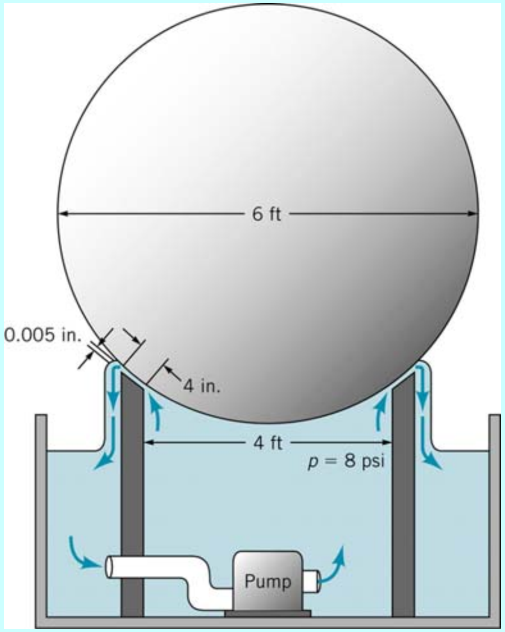
$$f_1(x) = \left(\frac{\partial p}{\partial x}\right)x + p_0$$

where p_0 is a reference pressure at $x=y=0$

Thus the pressure variation throughout the fluid can be obtained from

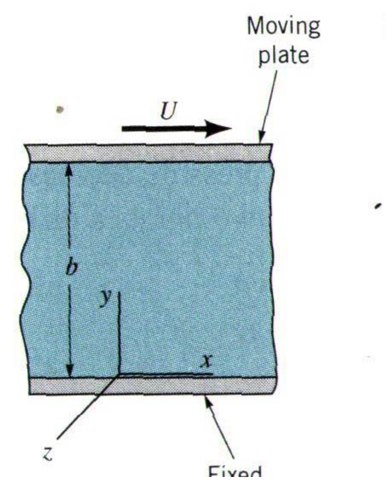
$$p = -\rho g y + \left(\frac{\partial p}{\partial x}\right)x + p_0$$

- The above analysis is valid for $\mathrm{Re} = \frac{\rho V 2h}{\mu}$ remains below about 1400



Problem 6.88: 10 tons on 8psi

6.9.2 Couette Flow



- Therefore

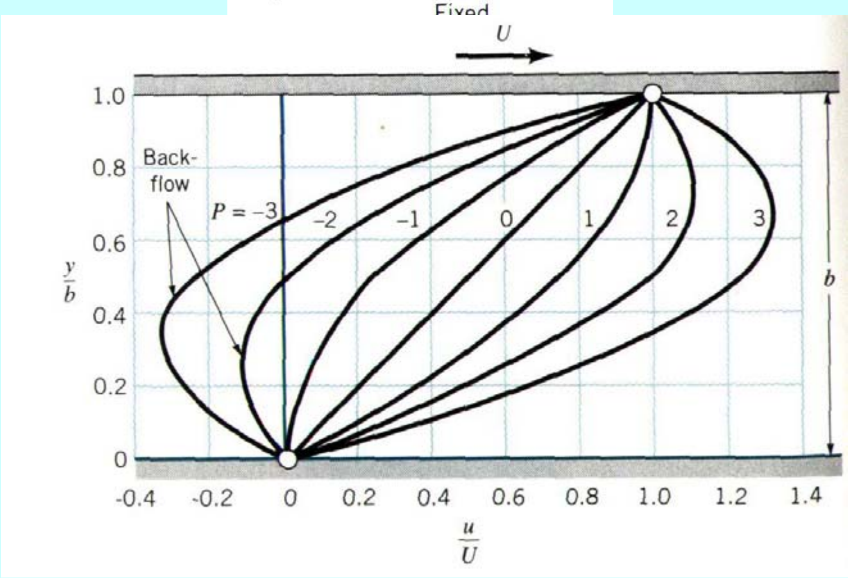
$$u = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)y^2 + C_1 y + C_2$$

boundary conditions
u=o at y=0, u=U at y=b

$$u = U\frac{b}{y} + \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)\left(y^2 - by\right)$$

or in dimensionless form

$$\frac{u}{U} = \frac{y}{b} - \frac{b^2}{2\mu U}\left(\frac{\partial p}{\partial x}\right)\left(\frac{y}{b}\right)\left(1 - \frac{y}{b}\right)$$



The actual velocity profile will depend on the dimensionless parameter

$$P = -\frac{b^2}{2\mu U}\left(\frac{\partial p}{\partial x}\right)$$

This type of flow is called **Couette flow**.

Couette flow

- The simplest type of Couette flow is one for which the pressure gradient is zero i.e. the fluid motion is caused by the fluid being dragged along by the moving boundary.

$$\frac{\partial p}{\partial x} = 0$$

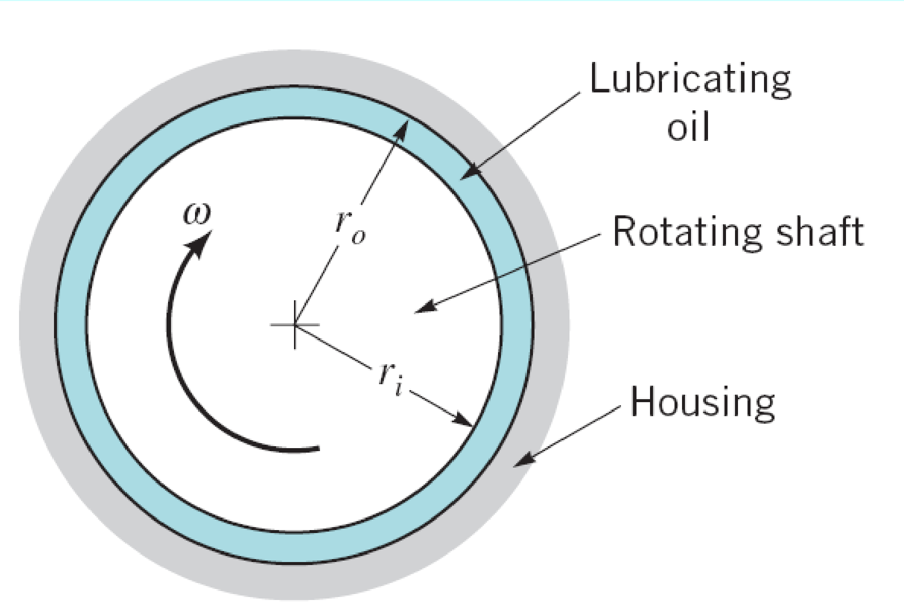
Thus
$$u = U \frac{y}{b}$$

which indicates that the velocity varies linearly between the two plates.

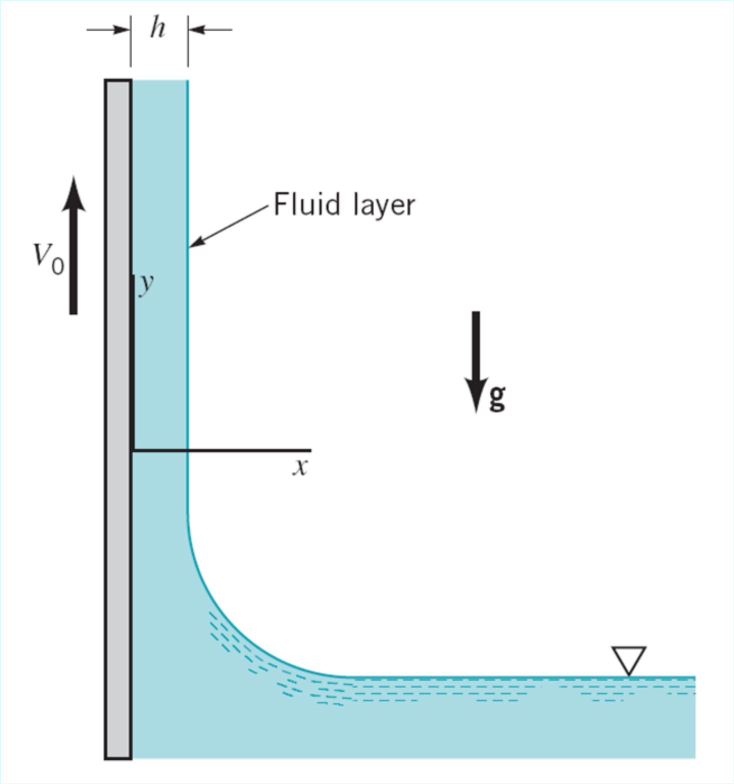
- e.g. ： Journal bearing

$$r_o - r_i << r_i$$

The flow in an unloaded journal bearing might be approximated by this simple Couette flow.



Example 6.9



$$u = w = 0 \qquad \frac{\partial \upsilon}{\partial y} = 0 \qquad \upsilon = \upsilon(x)$$

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial z} = 0$$

$$x = h \quad p = atmospheric \;\; pressure$$

$$\therefore \frac{dp}{dx} = 0 \qquad \therefore \frac{dp}{dz} = 0$$

Therefore

$$0 = -\rho g + \mu \frac{d^2 \upsilon}{dx^2}$$

$$\frac{d^2 \upsilon}{dx^2} = \frac{\gamma}{\mu}$$

$$\frac{d\upsilon}{dx} = \frac{\gamma}{\mu} x + C_1$$

on the film surface x=h, we assume that the shearing stress is zero

$$\tau_{xy} = \mu \left(\frac{d\upsilon}{dx} \right) \qquad \tau_{xy} = 0 \;\; at \;\; x = h$$

$$C_1 = -\frac{\gamma h}{\mu}$$

2nd integration

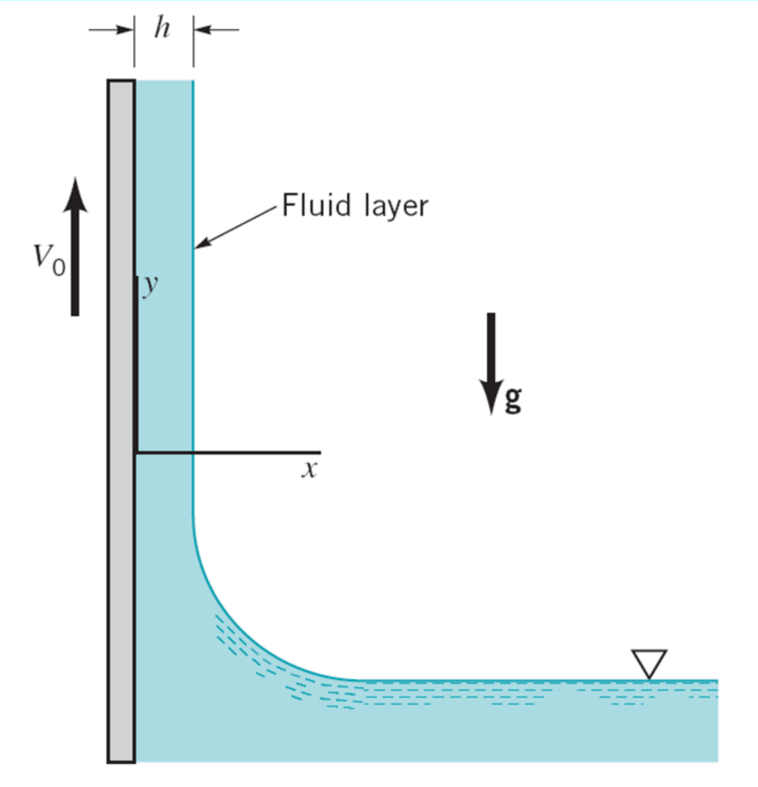
$$\upsilon = \frac{\gamma}{2\mu}x^2 - \frac{\gamma h}{\mu}x + C_2$$

$$x = 0 \quad \upsilon = V_0 \quad \therefore C_2 = V_0$$

$$\upsilon = \frac{\gamma}{2\mu}x^2 - \frac{\gamma h}{\mu}x + V_0$$

$$q = \int_0^h \upsilon \, dx = \int_0^h \left(\frac{\gamma}{2\mu}x^2 - \frac{\gamma h}{\mu}x + V_0 \right) dx$$

$$q = V_0 h - \frac{\gamma h^3}{3\mu}$$



The average film velocity

$$V = \frac{q}{h} = V_0 - \frac{\gamma h^2}{3\mu}$$

Only if $V_0 > \frac{\gamma h^2}{3\mu}$, will there be a net upward flow of liquid.

Q: Do you find anything weird in this problem?

6.9.3 Steady, Laminar flow in Circular Tubes

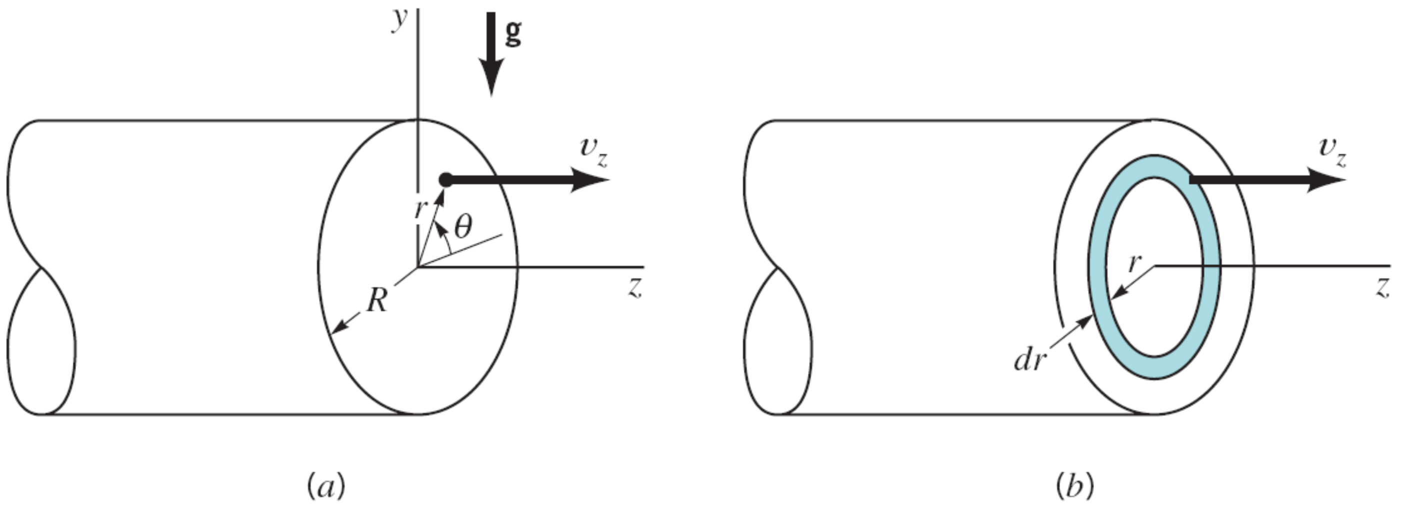
- Hagen–Poiseuille flow or Poiseuille flow
  steady, laminar flow through a straight circular tube of constant cross section
- Consider the flow through a horizontal circular tube of radius *R*

Assume the flow is parallel

$$v_r = v_\theta = 0$$

$$\frac{\partial v_z}{\partial z} = 0$$

$$\therefore v_z = v_z(r)$$



Steady, Laminar flow in Circular Tubes

Thus

$$0 = -\rho g \cos\theta - \frac{1}{r}\frac{\partial p}{\partial \theta} \qquad g_r = -g\sin\theta$$

$$0 = -\rho g \sin\theta - \frac{\partial p}{\partial r} \qquad g_\theta = -g\cos\theta$$

$$0 = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)\right]$$

Integration of equations in the *r* and θ directions

$$p = -\rho g r \sin\theta + f_1(z)$$

$$= -\rho g y + f_1(z)$$

which indicate that the pressure is hydrostatically distributed at any particular cross section and the *z* component of the pressure gradient, $\partial p/\partial z$, is not a function of *r* or θ.

Steady, Laminar flow in Circular Tubes

- the equation of motion in the z direction

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)=\frac{1}{\mu}\frac{\partial p}{\partial z}$$

$$r\frac{\partial v_z}{\partial r}=\frac{1}{2\mu}\left(\frac{\partial p}{\partial z}\right)r^2+C_1$$

$$v_z=\frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)r^2+C_1\ln r+C_2$$

- Boundary conditions

At $r=0$, v_z is finite at the center of the tube, thus $C_1=0$

At $r=R$, $v_z=0$, then $C_2=-\frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)R^2$

Thus the velocity distribution becomes,

$$v_z=\frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)\left(r^2-R^2\right)$$

That is, at any cross section, the velocity distribution is *parabolic*.

Steady, Laminar flow in Circular Tubes

- Volume flow rate

$$dQ = v_z(2\pi r)dr$$

$$Q = 2\pi\int_0^R v_z r dr = 2\pi\int_0^R \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)(r^2 - R^2) r dr = -\frac{\pi R^4}{8\mu}\frac{\partial p}{\partial z}$$

Let $\frac{\Delta p}{\ell} = -\frac{\partial p}{\partial z}$, then $Q = \frac{\pi R^4 \Delta p}{8\mu\ell}$ $\longleftarrow$ Poiseuille's law

- mean velocity

$$V = \frac{Q}{\pi R^2} = \frac{R^2 \Delta P}{8\mu\ell}$$

- maximum velocity

$$v_{\max} = \frac{R^2}{4\mu}\left(\frac{\partial p}{\partial z}\right) = \frac{R^2 \Delta p}{4\mu\ell}$$ so $$v_{\max} = 2V$$

- the velocity distribution in terms of $v_{\max}$

$$\frac{v_z}{v_{\max}} = 1 - \left(\frac{r}{R}\right)^2$$

6.9.4 Steady, Axial, Laminar Flow in an Annulus

$$v_z = \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right) r^2 + C_1 \ln r + C_2$$

- B.Cs：$v_z=0$ at $r=r_o$ and $r=r_i$

thus $$v_z = \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)\left[r^2 - r_0^2 + \frac{r_i^2 - r_0^2}{\ln\left(r_0/r_i\right)}\ln\left(r_0/r_i\right)\right]$$

- volume rate of flow

$$Q = \int_{r_i}^{r_0} v_z 2\pi r dr = -\frac{\pi}{8\mu}\left(\frac{\partial p}{\partial z}\right)\left[r_o^4 - r_i^4 - \frac{\left(r_o^2 - r_i^2\right)^2}{\ln\left(r_0/r_i\right)}\right]$$

$$= -\frac{\pi\Delta p}{8\mu\ell}\left[r_o^4 - r_i^4 - \frac{\left(r_o^2 - r_i^2\right)^2}{\ln\left(r_0/r_i\right)}\right]$$

The maximum velocity occur at the $r = r_m, \quad \partial v_z / \partial r = 0$

$$r_m = \left[\frac{r_o^2 - r_i^2}{2 \ln \left(r_0 / r_i \right)} \right]^{1/2}$$

The maximum velocity does not occur at the mid point of the annulus space, but rather it occurs nearer the inner cylinder.

- To determine Reynolds number, it is common practice to use an effective diameter “hydraulic diameter” for on circular tubes.

$$D_h = \frac{4 \times \text{cross-sctional area}}{\text{wetted perimeter}}$$

Thus the flow will remain laminar if $R_e = \dfrac{\rho D_h V}{\mu}$ remains below 2100.

6.10 Other Aspects of Differential Analysis

$$\nabla \bullet \vec{V} = 0$$

$$\rho\left(\frac{\partial \vec{V}}{\partial t} + \vec{V} \bullet \nabla \vec{V}\right) = -\nabla p + \rho g + \mu \nabla^2 \vec{V}$$

- The solutions of the equations and not readily available.

6.10.1 Numerical Methods

- Finite difference method
- Finite element (or finite volume) method
- Boundary element method

V6.15 CFD example